

Curvilinear Coordinates

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B1. Cylindrical Polar Coordinates

The coordinates are (R, θ, x) , where θ is the azimuthal angle (see Figure 3.1b, where φ is used instead of θ). The equations are presented assuming ψ is a scalar, and

$$\mathbf{u} = \mathbf{i}_R u_R + \mathbf{i}_\theta u_\theta + \mathbf{i}_x u_x,$$

where $\mathbf{i}_R$, $\mathbf{i}_\theta$, and $\mathbf{i}_x$ are the local unit vectors at a point.

Gradient of a scalar

$$\nabla \psi = \mathbf{i}_R \frac{\partial \psi}{\partial R} + \frac{\mathbf{i}_\theta}{R} \frac{\partial \psi}{\partial \theta} + \mathbf{i}_x \frac{\partial \psi}{\partial x}.$$

Laplacian of a scalar

$$\nabla^2 \psi = \frac{1}{R} \frac{\partial}{\partial R} \left(R \frac{\partial \psi}{\partial R} \right) + \frac{1}{R^2} \frac{\partial^2 \psi}{\partial \theta^2} + \frac{\partial^2 \psi}{\partial x^2}.$$

Divergence of a vector

$$\nabla \cdot \mathbf{u} = \frac{1}{R} \frac{\partial (R u_R)}{\partial R} + \frac{1}{R} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_x}{\partial x}.$$

Curl of a vector

$$\nabla \times \mathbf{u} = \mathbf{i}_R \left(\frac{1}{R} \frac{\partial u_x}{\partial \theta} - \frac{\partial u_\theta}{\partial x} \right) + \mathbf{i}_\theta \left(\frac{\partial u_R}{\partial x} - \frac{\partial u_x}{\partial R} \right) + \mathbf{i}_x \left[\frac{1}{R} \frac{\partial (R u_\theta)}{\partial R} - \frac{1}{R} \frac{\partial u_R}{\partial \theta} \right].$$

Laplacian of a vector

$$\nabla^2 \mathbf{u} = \mathbf{i}_R \left(\nabla^2 u_R - \frac{u_R}{R^2} - \frac{2}{R^2} \frac{\partial u_\theta}{\partial \theta} \right) + \mathbf{i}_\theta \left(\nabla^2 u_\theta + \frac{2}{R^2} \frac{\partial u_R}{\partial \theta} - \frac{u_\theta}{R^2} \right) + \mathbf{i}_x \nabla^2 u_x.$$

Strain rate and viscous stress (for incompressible form $\sigma_{ij} = 2\mu e_{ij}$)

$$\begin{aligned}
 e_{RR} &= \frac{\partial u_R}{\partial R} = \frac{1}{2\mu} \sigma_{RR}, \\
 e_{\theta\theta} &= \frac{1}{R} \frac{\partial u_\theta}{\partial \theta} + \frac{u_R}{R} = \frac{1}{2\mu} \sigma_{\theta\theta}, \\
 e_{xx} &= \frac{\partial u_x}{\partial x} = \frac{1}{2\mu} \sigma_{xx}, \\
 e_{R\theta} &= \frac{R}{2} \frac{\partial}{\partial R} \left(\frac{u_\theta}{R} \right) + \frac{1}{2R} \frac{\partial u_R}{\partial \theta} = \frac{1}{2\mu} \sigma_{R\theta}, \\
 e_{\theta x} &= \frac{1}{2R} \frac{\partial u_x}{\partial \theta} + \frac{1}{2} \frac{\partial u_\theta}{\partial x} = \frac{1}{2\mu} \sigma_{\theta x}, \\
 e_{xR} &= \frac{1}{2} \frac{\partial u_R}{\partial x} + \frac{1}{2} \frac{\partial u_x}{\partial R} = \frac{1}{2\mu} \sigma_{xR}.
 \end{aligned}$$

Vorticity ($\boldsymbol{\omega} = \nabla \times \mathbf{u}$)

$$\begin{aligned}
 \omega_R &= \frac{1}{R} \frac{\partial u_x}{\partial \theta} - \frac{\partial u_\theta}{\partial x}, \\
 \omega_\theta &= \frac{\partial u_R}{\partial x} - \frac{\partial u_x}{\partial R}, \\
 \omega_x &= \frac{1}{R} \frac{\partial}{\partial R} (Ru_\theta) - \frac{1}{R} \frac{\partial u_R}{\partial \theta}.
 \end{aligned}$$

Equation of continuity

$$\frac{\partial \rho}{\partial t} + \frac{1}{R} \frac{\partial}{\partial R} (\rho R u_R) + \frac{1}{R} \frac{\partial}{\partial \theta} (\rho u_\theta) + \frac{\partial}{\partial x} (\rho u_x) = 0.$$

Navier–Stokes equations with constant ρ and ν , and no body force

$$\begin{aligned}
 \frac{\partial u_R}{\partial t} + (\mathbf{u} \cdot \nabla) u_R - \frac{u_\theta^2}{R} &= -\frac{1}{\rho} \frac{\partial p}{\partial R} + \nu \left(\nabla^2 u_R - \frac{u_R}{R^2} - \frac{2}{R^2} \frac{\partial u_\theta}{\partial \theta} \right), \\
 \frac{\partial u_\theta}{\partial t} + (\mathbf{u} \cdot \nabla) u_\theta + \frac{u_R u_\theta}{R} &= -\frac{1}{\rho R} \frac{\partial p}{\partial \theta} + \nu \left(\nabla^2 u_\theta + \frac{2}{R^2} \frac{\partial u_R}{\partial \theta} - \frac{u_\theta}{R^2} \right), \\
 \frac{\partial u_x}{\partial t} + (\mathbf{u} \cdot \nabla) u_x &= -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \nabla^2 u_x,
 \end{aligned}$$

where

$$\begin{aligned}
 \mathbf{u} \cdot \nabla &= u_R \frac{\partial}{\partial R} + \frac{u_\theta}{R} \frac{\partial}{\partial \theta} + u_x \frac{\partial}{\partial x}, \\
 \nabla^2 &= \frac{1}{R} \frac{\partial}{\partial R} \left(R \frac{\partial}{\partial R} \right) + \frac{1}{R^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial x^2}.
 \end{aligned}$$

B2. Plane Polar Coordinates

The plane polar coordinates are (r, θ) , where r is the distance from the origin (Figure 3.1a). The equations for plane polar coordinates can be obtained from those of the cylindrical coordinates presented in Section B1, replacing R by r and suppressing all components and derivatives in the axial direction x . Some of the expressions are repeated here because of their frequent occurrence.

Strain rate and viscous stress (for incompressible form $\sigma_{ij} = 2\mu e_{ij}$)

$$\begin{aligned} e_{rr} &= \frac{\partial u_r}{\partial r} = \frac{1}{2\mu} \sigma_{rr}, \\ e_{\theta\theta} &= \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} = \frac{1}{2\mu} \sigma_{\theta\theta}, \\ e_{r\theta} &= \frac{r}{2} \frac{\partial}{\partial r} \left(\frac{u_\theta}{r} \right) + \frac{1}{2r} \frac{\partial u_r}{\partial \theta} = \frac{1}{2\mu} \sigma_{r\theta}. \end{aligned}$$

Vorticity

$$\omega_z = \frac{1}{r} \frac{\partial}{\partial r} (ru_\theta) - \frac{1}{r} \frac{\partial u_r}{\partial \theta}.$$

Equation of continuity

$$\frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (\rho r u_r) + \frac{1}{r} \frac{\partial}{\partial \theta} (\rho u_\theta) = 0.$$

Navier–Stokes equations with constant ρ and ν , and no body force

$$\begin{aligned} \frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta^2}{r} &= -\frac{1}{\rho} \frac{\partial p}{\partial r} + \nu \left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2} \frac{\partial u_\theta}{\partial \theta} \right), \\ \frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r u_\theta}{r} &= -\frac{1}{\rho r} \frac{\partial p}{\partial \theta} + \nu \left(\nabla^2 u_\theta + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2} \right), \end{aligned}$$

where

$$\nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}.$$

B3. Spherical Polar Coordinates

The spherical polar coordinates used are (r, θ, φ) , where φ is the azimuthal angle (Figure 3.1c). Equations are presented assuming ψ is a scalar, and

$$\mathbf{u} = \mathbf{i}_r u_r + \mathbf{i}_\theta u_\theta + \mathbf{i}_\varphi u_\varphi,$$

where $\mathbf{i}_r$, $\mathbf{i}_\theta$, and $\mathbf{i}_\varphi$ are the local unit vectors at a point.

Gradient of a scalar

$$\nabla \psi = \mathbf{i}_r \frac{\partial \psi}{\partial r} + \mathbf{i}_\theta \frac{1}{r} \frac{\partial \psi}{\partial \theta} + \mathbf{i}_\varphi \frac{1}{r \sin \theta} \frac{\partial \psi}{\partial \varphi}.$$

Laplacian of a scalar

$$\nabla^2 \psi = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \psi}{\partial \varphi^2}.$$

Divergence of a vector

$$\nabla \cdot \mathbf{u} = \frac{1}{r^2} \frac{\partial (r^2 u_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial (u_\theta \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial u_\varphi}{\partial \varphi}.$$

Curl of a vector

$$\begin{aligned} \nabla \times \mathbf{u} = & \frac{\mathbf{i}_r}{r \sin \theta} \left[\frac{\partial (u_\varphi \sin \theta)}{\partial \theta} - \frac{\partial u_\theta}{\partial \varphi} \right] + \frac{\mathbf{i}_\theta}{r} \left[\frac{1}{\sin \theta} \frac{\partial u_r}{\partial \varphi} - \frac{\partial (r u_\varphi)}{\partial r} \right] \\ & + \frac{\mathbf{i}_\varphi}{r} \left[\frac{\partial (r u_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta} \right]. \end{aligned}$$

Laplacian of a vector

$$\begin{aligned} \nabla^2 \mathbf{u} = & \mathbf{i}_r \left[\nabla^2 u_r - \frac{2u_r}{r^2} - \frac{2}{r^2 \sin \theta} \frac{\partial (u_\theta \sin \theta)}{\partial \theta} - \frac{2}{r^2 \sin \theta} \frac{\partial u_\varphi}{\partial \varphi} \right] \\ & + \mathbf{i}_\theta \left[\nabla^2 u_\theta + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2 \sin^2 \theta} - \frac{2 \cos \theta}{r^2 \sin^2 \theta} \frac{\partial u_\varphi}{\partial \varphi} \right] \\ & + \mathbf{i}_\varphi \left[\nabla^2 u_\varphi + \frac{2}{r^2 \sin \theta} \frac{\partial u_r}{\partial \varphi} + \frac{2 \cos \theta}{r^2 \sin^2 \theta} \frac{\partial u_\theta}{\partial \varphi} - \frac{u_\varphi}{r^2 \sin^2 \theta} \right]. \end{aligned}$$

Strain rate and viscous stress (for incompressible form $\sigma_{ij} = 2\mu e_{ij}$)

$$\begin{aligned} e_{rr} &= \frac{\partial u_r}{\partial r} = \frac{1}{2\mu} \sigma_{rr}, \\ e_{\theta\theta} &= \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} = \frac{1}{2\mu} \sigma_{\theta\theta}, \\ e_{\varphi\varphi} &= \frac{1}{r \sin \theta} \frac{\partial u_\varphi}{\partial \varphi} + \frac{u_r}{r} + \frac{u_\theta \cot \theta}{r} = \frac{1}{2\mu} \sigma_{\varphi\varphi}, \\ e_{\theta\varphi} &= \frac{\sin \theta}{2r} \frac{\partial}{\partial \theta} \left(\frac{u_\varphi}{\sin \theta} \right) + \frac{1}{2r \sin \theta} \frac{\partial u_\theta}{\partial \varphi} = \frac{1}{2\mu} \sigma_{\theta\varphi}, \\ e_{\varphi r} &= \frac{1}{2r \sin \theta} \frac{\partial u_r}{\partial \varphi} + \frac{r}{2} \frac{\partial}{\partial r} \left(\frac{u_\varphi}{r} \right) = \frac{1}{2\mu} \sigma_{\varphi r}, \\ e_{r\theta} &= \frac{r}{2} \frac{\partial}{\partial r} \left(\frac{u_\theta}{r} \right) + \frac{1}{2r} \frac{\partial u_r}{\partial \theta} = \frac{1}{2\mu} \sigma_{r\theta}. \end{aligned}$$

Vorticity

$$\begin{aligned}\omega_r &= \frac{1}{r \sin \theta} \left[\frac{\partial}{\partial \theta} (u_\varphi \sin \theta) - \frac{\partial u_\theta}{\partial \varphi} \right], \\ \omega_\theta &= \frac{1}{r} \left[\frac{1}{\sin \theta} \frac{\partial u_r}{\partial \varphi} - \frac{\partial (ru_\varphi)}{\partial r} \right], \\ \omega_\varphi &= \frac{1}{r} \left[\frac{\partial}{\partial r} (ru_\theta) - \frac{\partial u_r}{\partial \theta} \right].\end{aligned}$$

Equation of continuity

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} (\rho r^2 u_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\rho u_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} (\rho u_\varphi) = 0.$$

Navier–Stokes equations with constant ρ and ν , and no body force

$$\begin{aligned}\frac{\partial u_r}{\partial t} + (\mathbf{u} \cdot \nabla) u_r - \frac{u_\theta^2 + u_\varphi^2}{r} \\ = -\frac{1}{\rho} \frac{\partial p}{\partial r} + \nu \left[\nabla^2 u_r - \frac{2u_r}{r^2} - \frac{2}{r^2 \sin \theta} \frac{\partial (u_\theta \sin \theta)}{\partial \theta} - \frac{2}{r^2 \sin \theta} \frac{\partial u_\varphi}{\partial \varphi} \right], \\ \frac{\partial u_\theta}{\partial t} + (\mathbf{u} \cdot \nabla) u_\theta + \frac{u_r u_\theta}{r} - \frac{u_\varphi^2 \cot \theta}{r} \\ = -\frac{1}{\rho r} \frac{\partial p}{\partial \theta} + \nu \left[\nabla^2 u_\theta + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2 \sin^2 \theta} - \frac{2 \cos \theta}{r^2 \sin^2 \theta} \frac{\partial u_\varphi}{\partial \varphi} \right], \\ \frac{\partial u_\varphi}{\partial t} + (\mathbf{u} \cdot \nabla) u_\varphi + \frac{u_\varphi u_r}{r} + \frac{u_\theta u_\varphi \cot \theta}{r} \\ = -\frac{1}{\rho r \sin \theta} \frac{\partial p}{\partial \varphi} + \nu \left[\nabla^2 u_\varphi + \frac{2}{r^2 \sin \theta} \frac{\partial u_r}{\partial \varphi} + \frac{2 \cos \theta}{r^2 \sin^2 \theta} \frac{\partial u_\theta}{\partial \varphi} - \frac{u_\varphi}{r^2 \sin^2 \theta} \right],\end{aligned}$$

where

$$\begin{aligned}\mathbf{u} \cdot \nabla &= u_r \frac{\partial}{\partial r} + \frac{u_\theta}{r} \frac{\partial}{\partial \theta} + \frac{u_\varphi}{r \sin \theta} \frac{\partial}{\partial \varphi}, \\ \nabla^2 &= \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \varphi^2}.\end{aligned}$$