FÍSICA TEÓRICA 1 – 1er. Cuatrimestre de 2025

APUNTE DE FÓRMULAS PARA SEPARACIÓN DE VARIABLES

Coordenadas cartesianas

Ortogonalidad:

$$\int_0^a dx\,\sin(k_{n'}x)\sin(k_n x) = \frac{a}{2}\delta_{nn'}\,, \quad k_n = n\pi/a$$

$$\int_0^{+\infty} dx\,\sin(k'x)\sin(kx) = \frac{\pi}{2}\,\delta(k-k')\,, \quad k>0$$

$$\int_{-\infty}^{+\infty} dx\,\sin(k'x)\sin(kx) = \pi\,\delta(k-k')\,, \quad k>0$$

$$\int_{-\infty}^{+\infty} dx\,\cos(k'x)\cos(kx) = \pi\,\delta(k-k')\,, \quad k>0$$

$$\int_{-\infty}^{\infty} dx\, e^{i(k-k')x} = 2\pi\delta(k-k')\,, \quad k\in\mathbb{R}$$

Coordenadas esféricas

Coordenadas y etiquetas: $\theta\in[0,\pi]$ $\quad\varphi\in[0,2\pi]$ $\quad l\in\mathbb{N}_0\,; -l\le m\le l\,, m\in\mathbb{Z}$

Armónicos esféricos: $Y_{lm}(\theta,\varphi)=\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{im\varphi}$

Funciones asociadas de Legendre: $P_l^m(x)=(-1)^m(1-x^2)^{m/2}\frac{d^m}{dx^m}P_l(x) \quad (m\geqslant 0)$

Polinomios de Legendre: $P_l(x)=\frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l$

Ortogonalidad: $(x=\cos\theta)$

$\int_0^{2\pi}d\varphi\int_0^{\pi}\sin(\theta)\,d\theta\,Y^*_{l'm'}(\theta,\varphi)Y_{lm}(\theta,\varphi)=\delta_{l'l}\delta_{m'm}$ $\qquad$ $\int_{-1}^1 dx\,P_l(x)P_{l'}(x)=\frac{2}{2l+1}\delta_{l'l}$

$\int_{-1}^1 dx\,P_{l'}^m(x)P_l^m(x)=\frac{2}{2l+1}\frac{(l+m)!}{(l-m)!}\delta_{l'l}$ $\qquad$ $\int_0^{2\pi}d\varphi\,e^{im\varphi}e^{-im'\varphi}=2\pi\delta_{m'm}$

Completitud:

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l}Y^*_{lm}(\theta',\varphi')Y_{lm}(\theta,\varphi)=\delta(\varphi-\varphi')\delta(\cos\theta-\cos\theta')$$

Teorema de adición de armónicos esféricos: $[\cos\gamma=\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\varphi-\varphi')\,]$

$$P_l(\cos\gamma)=\frac{4\pi}{2l+1}\sum_{m=-l}^{l}Y^*_{lm}(\theta',\varphi')Y_{lm}(\theta,\varphi)$$

Propiedades:

$$Y_{l,-m}(\theta,\varphi) = (-1)^m Y^*_{lm}(\theta,\varphi) \qquad P_l^{-m}(x) = (-1)^m \tfrac{(l-m)!}{(l+m)!} P_l^m(x)$$

Recurrencia: $\frac{dP_{l+1}}{dx}(x) - \frac{dP_{l-1}}{dx}(x) = (2l+1)P_l(x) \qquad P_l(-x) = (-1)^l P_l(x)$

$$P_l(0) = \begin{cases} 0 & \text{si } \; l \; \text{ es impar} \\ \frac{(-1)^{l/2}(l-1)!!}{2^{l/2}\left(\frac{l}{2}\right)!} & \text{si } \; l \; \text{ es par} \end{cases} \qquad P_l(1) = 1$$

Primeros polinomios de Legendre:

$$P_0(x) = 1 \qquad P_1(x) = x \qquad P_2(x) = \tfrac{1}{2}(3x^2-1) \qquad P_3(x) = \tfrac{1}{2}(5x^3-3x) \qquad P_4(x) = \tfrac{1}{8}(35x^4 - 30x^2 + 3)$$

Primeros armónicos esféricos:

$$Y_{00}(\theta,\varphi) = \tfrac{1}{\sqrt{4\pi}}$$
$$Y_{10}(\theta,\varphi) = \sqrt{\tfrac{3}{4\pi}}\,\cos\theta \qquad Y_{11}(\theta,\varphi) = -\sqrt{\tfrac{3}{8\pi}}\,\sin\theta\, e^{i\varphi}$$
$$Y_{20}(\theta,\varphi) = \sqrt{\tfrac{5}{4\pi}}\,\left(\tfrac{3}{2}\cos^2\theta - \tfrac{1}{2}\right) \qquad Y_{21}(\theta,\varphi) = -\sqrt{\tfrac{15}{8\pi}}\,\sin\theta\,\cos\theta\, e^{i\varphi} \qquad Y_{22}(\theta,\varphi) = \tfrac{1}{4}\sqrt{\tfrac{15}{2\pi}}\,\sin^2\theta\, e^{2i\varphi}$$

Coordenadas cilíndricas

Coordenadas y etiquetas:

$$\varphi \in [0,2\pi] \quad \nu \in \mathbb{N}_0$$
$$k \geqslant 0$$

De la separación de variables: $\Phi \sim Q(\varphi)Z(z)R(\rho)$

$$\begin{aligned} Q''(\varphi) &= -\beta\, Q(\varphi) \\ Z''(z) &= \lambda\, Z(z) \\ R''(\rho) + R'(\rho)/\rho &= \left(\beta/\rho^2 - \lambda\right) R(\rho) \end{aligned}$$

λ	β	$Q(\varphi)$	$Z(z)$	$R(\rho)$
k^2	$\nu^2 = 0$	$1, \varphi$	$e^{\pm kz}$	$J_\nu(k\rho), N_\nu(k\rho)$
	ν^2	$e^{\pm i\nu\varphi}$		
0	0	$1, \varphi$	$1, z$	$1, \ln(\rho)$
	ν^2	$e^{\pm i\nu\varphi}$		$\rho^{\pm\nu}$
$-k^2$	$\nu^2 = 0$	$1, \varphi$	$e^{\pm ikz}$	$I_\nu(k\rho), K_\nu(k\rho)$
	ν^2	$e^{\pm i\nu\varphi}$		

Para cada ν, las funciones $\{J_\nu(k\rho), N_\nu(k\rho)\}$ son independientes y forman base si:

intervalo finito: $\rho \in [0;a] \;\rightarrow\; k \equiv k_{\nu n} = \chi_{\nu n}/a$
intervalo infinito: $\rho \in [0;\infty) \rightarrow\; k \in [0;\infty)$

con $\{\chi_{\nu n}\}_{n\in\mathbb{N}}$ las infinitas raíces de $J_\nu(x)$.

Ortogonalidad de las funciones de Bessel J_ν:

$$\int_0^a d\rho\, \rho\, J_\nu(x_{\nu n'}\, \rho/a) J_\nu(x_{\nu n}\, \rho/a) = \frac{a^2}{2} \left[J_{|\nu|+1}(x_{\nu n})\right]^2 \delta_{nn'}$$

$$\int_0^\infty d\rho\, \rho\, J_\nu(k\rho) J_\nu(k'\rho) = \frac{\delta(k-k')}{k}$$

Propiedades: $(x > 0)$

$$J_0(0) = 1\,, \quad J_{\nu\, \neq 0}(0) = 0\,, \quad x \ll 1: \quad J_\nu(x) \sim \tfrac{1}{\Gamma(\nu+1)} \left(\tfrac{x}{2}\right)^\nu \quad x \gg \nu^2: \quad J_\nu(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$$

$$N_\nu(x \to 0) \to \infty \qquad \nu \in \mathbb{N}: \quad J_{-\nu}(x) = (-1)^\nu J_\nu(x)\,, \qquad \int dx\, x^\nu J_{\nu-1}(x) = x^\nu J_\nu(x)$$

Fórmulas de recurrencia:

$$\Omega_{\nu-1}(x) + \Omega_{\nu+1}(x) = \frac{2\nu}{x} \Omega_\nu(x)$$

$$\Omega_{\nu-1}(x) - \Omega_{\nu+1}(x) = 2\frac{d\Omega_\nu}{dx}(x)$$

Donde Ω_ν representa a cualquiera de las funciones J_ν, N_ν, $H_\nu^{(1)} = J_\nu + iN_\nu$ y $H_\nu^{(2)} = J_\nu - iN_\nu$.

Para las modificadas:

$$I_0(0) = 1\,,\ I_{\nu>0}(0) = 0\,,\ I_\nu(x \to \infty) \to \infty \qquad K_\nu(x \to 0) \to \infty\,,\ K_\nu(x \to \infty) = 0$$

$$I_{-\nu} = I_\nu\,, \qquad K_{-\nu} = K_\nu$$

Wronskiano: $W[K_\nu, I_\nu](x) \equiv K_\nu(x) I'_\nu(x) - K'_\nu(x) I_\nu(x) = \frac{1}{x}$

Sobre la delta de Dirac $\delta(\rho - \rho')$ en cilíndricas, notar que si $\rho' = 0$, vale:

$$\int_0^{\epsilon>0} d\rho\, \delta(\rho) = 1$$

Funciones trigonométricas: definiciones y propiedades

$$\tan(x) = \tfrac{\sin(x)}{\cos(x)} \qquad \sec(x) = \tfrac{1}{\cos(x)} \qquad \coth(x) = \tfrac{\cos(x)}{\sin(x)} \qquad \text{cosec}(x) = \tfrac{1}{\sin(x)}$$

$$\sin^2(x) + \cos^2(x) = 1 \qquad \sin^2(x) = \tfrac{1}{2}(1 - \cos(2x)) \qquad \cos^2(x) = \tfrac{1}{2}(1 + \cos(2x))$$

$$2\sin(x)\sin(y) = \cos(x-y) - \cos(x+y) \qquad \sin(x \pm y) = \sin(x)\cos(y) \pm \sin(y)\cos(x)$$

$$2\cos(x)\cos(y) = \cos(x-y) + \cos(x+y) \qquad \cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$$

$$2\sin(x)\sin(y) = \sin(x-y) + \sin(x+y) \qquad \tan(x \pm y) = \tfrac{\tan(x) \pm \tan(y)}{1 \mp \tan(x)\tan(y)}$$

Funciones hiperbólicas: definiciones y propiedades

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \tanh(x) = \frac{\sinh(x)}{\cosh(x)}$$

$$\text{cosech}(x) = \frac{1}{\sinh(x)} \qquad \text{sech}(x) = \frac{1}{\cosh(x)} \qquad \text{cotanh}(x) = \frac{\cosh(x)}{\sinh(x)}$$

$$\cosh^2(x) - \sinh^2(x) = 1 \qquad \sinh(x \pm y) = \sinh(x)\cosh(y) \pm \sinh(y)\cosh(x)$$

$$\tanh(x \pm y) = \frac{\tanh(x) \pm \tanh(y)}{1 \pm \tanh(x)\tanh(y)} \qquad \cosh(x \pm y) = \cosh(x)\cosh(y) \pm \sinh(x)\sinh(y)$$