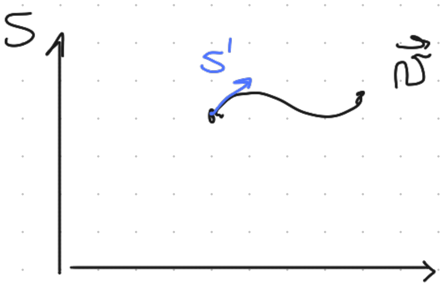
Dinámica Relativista.



$\vec{v}$ $\qquad$ $\frac{d\bar{r}}{dt}$

$\vec{u} = \vec{v}$.

$$ds^2 = c^2 d\tau^2 = c^2 dt - |d\bar{r}|^2 \Rightarrow d\tau^2 = dt^2 \underbrace{\left(1 - \frac{\left|\frac{d\bar{r}}{dt}\right|^2}{c^2}\right)}_{1 - \frac{v^2}{c^2} = \gamma^{-2}}$$

$$\Rightarrow \boxed{dt^2 = \gamma^2(v)\, d\tau^2}$$

↳ v de la partícula.

Reparametrizamos la trayectoria

$$x^\mu = \left(ct(\tau), \vec{r}(\tau)\right) \longrightarrow \text{Def: } u^\mu \equiv \frac{dx^\mu}{d\tau} = \left(c\frac{dt}{d\tau}, \frac{d\vec{x}}{dt}\frac{dt}{d\tau}\right)$$

$$\rightarrow \boxed{u^\mu = \gamma(v)\,(c, \vec{v})}$$

Def. el cuadri-impulso.

$$\boxed{p^\mu \equiv m\, u^\mu} = m\gamma(v)\,(c, \vec{v}). \rightarrow p^0 = \frac{m\gamma(v)c^2}{c} = \frac{\mathcal{E}}{c}$$

↳ $p^i = \gamma(v)\,\vec{p}$.

La dinámica está regida por:

$$\frac{dp^\mu}{d\tau} = \frac{q}{mc} F^{\mu\nu} p_\nu. \rightarrow$$

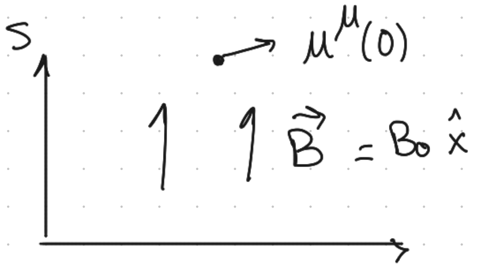
0: Ley de Joule.

i: Fuerza de Lorentz.

Para una dist. continua: $f^\mu = \frac{1}{c} F^{\mu\nu} J_\nu$, con

$J_\nu = (c\rho, -\vec{J})$

Problema 12.6



$$\frac{1}{m}\frac{dp^\mu}{d\tau} = \frac{q}{mc} \underbrace{F^{\mu\nu}}_{\neq} p_\nu \cdot \frac{1}{m}$$

$$\frac{du^\mu}{d\tau} = \frac{q}{mc} F^\mu{}_\nu \, u^\nu$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix} = \begin{pmatrix} \bar{\bar{0}} & \bar{\bar{0}} \\ \bar{\bar{0}} & \begin{matrix} 0 & B_x \\ -B_x & 0 \end{matrix} \end{pmatrix}$$

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$$F^\mu{}_\nu = \eta_{\nu\alpha} F^{\mu\alpha} = \eta_{\nu 0} F^{\mu 0} + \eta_{\nu 1} F^{\mu 1} + \eta_{\nu 2} F^{\mu 2} + \eta_{\nu 3} F^{\mu 3}.$$

$$\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1) \qquad \eta_{00} = 1, \quad \eta_{ii} = -1.$$

$$F^3{}_2 = \eta_{22} F^{32} = (-1)\, B_0 = -B_0$$

$$F^2{}_3 = \eta_{33} F^{23} = (-1)(-B_0) = B_0$$

$$F^\mu{}_\nu = B_0 \begin{pmatrix} \bar{\bar{0}} & \bar{\bar{0}} \\ \bar{\bar{0}} & \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} \end{pmatrix}$$

$\nu = 3$.

$$\frac{du^\mu}{d\tau} = \frac{q}{mc} F^\mu{}_\nu u^\nu = \frac{q}{mc}\left(F^\mu{}_0 u^0 + \dots \right)$$

- $\mu = 0 \rightarrow \dfrac{du^0}{d\tau} = 0 \Rightarrow u^0 = \text{cte} = \gamma(v)\, c \Rightarrow |v| = \text{cte}$

  $\gamma(v) \equiv \gamma.$

  $\rightarrow dt = \gamma \, d\tau.$

- $\mu = 1 \rightarrow \dfrac{du^1}{d\tau} = 0 \Rightarrow \overset{\text{cte}}{\gamma}\, \underbrace{v_x}_{\text{cte}} = \text{cte}. \Rightarrow v_x = v_x(0)$

- $\mu = 2,3.$

$$\begin{cases} \dfrac{du^2}{d\tau}\dfrac{1}{\gamma} = \dfrac{q}{mc} F^2{}_3 u^3 = -\dfrac{qB_0}{mc} u^3 \dfrac{1}{\gamma} \\[2ex] \dfrac{du^3}{d\tau}\dfrac{1}{\gamma} = \dfrac{q}{mc} F^3{}_2 u^2 = \dfrac{qB_0}{mc} u^2 \cdot \dfrac{1}{\gamma} \end{cases}$$

$$u^i = \gamma v_i$$

$$\begin{cases} \dfrac{dv_y}{d\tau} = -\omega\, v_z \\[2mm] \dfrac{dv_z}{d\tau} = \omega\, v_y \end{cases} \longrightarrow \frac{d^2 v_{2,3}}{d\tau^2} + \omega^2 v_{2,3} = 0$$

$$\Rightarrow \begin{cases} v_y(\tau) = A \sin(\omega\tau + \alpha) \\ v_z(\tau) = B \cos(\omega\tau + \beta) \end{cases} \longrightarrow \begin{cases} A\cos(\omega\tau+\alpha) = -B\cos(\omega\tau+\beta) \\ A\sin(\omega\tau+\alpha) = -B\sin(\omega\tau+\beta) \end{cases}$$

Dividiendo las ec.

$$\tan(\omega\tau+\alpha) = \tan(\omega\tau+\beta) \Rightarrow \alpha = \beta.$$

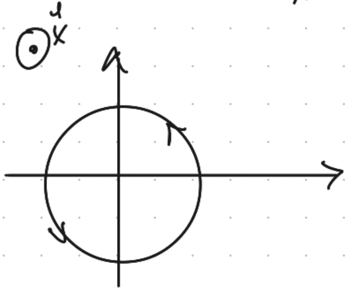
$$\Rightarrow A = -B.$$

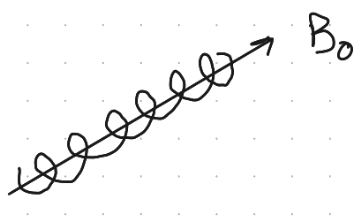
Ponemos CI: $\begin{cases} v_y(0) = 0 \\ v_z(0) = v_\varphi. \end{cases}$

$$\begin{cases} \sin\alpha = 0 \Rightarrow \alpha = 0 \\ B = v_\varphi. \end{cases} \longrightarrow \begin{cases} v_y(\tau) = -v_\varphi \sin(\omega\tau) \\ v_z(\tau) = v_\varphi \cos(\omega\tau). \end{cases}$$

$$\vec{v}(\tau) = v_x\,\hat{x} - v_\varphi \sin(\omega\tau)\,\hat{y} + v_\varphi \cos(\omega\tau)\,\hat{z}.$$

$$= v_x\,\hat{x} + v_\varphi\,\hat{\varphi}(\omega\tau).$$





Para recuperar la parametrización con el tiempo coordenado t.

$$\tau = \frac{t}{\gamma} \rightarrow \gamma(v) \rightarrow v = \sqrt{v_x^2 + v_\varphi^2}$$

$$\vec{v}(t) = v_x\,\hat{x} + v_\varphi\,\hat{\varphi}\left(\frac{\omega}{\gamma}t\right) \qquad \tilde{\omega} = \frac{\omega}{\gamma} < \omega \rightarrow \text{Redshift.}$$

Item c.

- $\vec{E}\cdot\vec{B}=0$. $\qquad |E|^2-|\vec{B}|^2 = \text{inv.}$

① $|\vec{E}|<|\vec{B}|$: $\exists S'$ tq $|\vec{E}|=0$. $\rightarrow$ item b.

② $|\vec{E}|>|\vec{B}|$: $\exists S'$ tq $|\vec{B}|=0$ $\rightarrow$ item a.

③ $|\vec{E}|=|\vec{B}|$ $\rightarrow$ Resolvamos este.

Las ecs. de mov.

$$\frac{du^\mu}{d\tau} = \frac{q}{mc} F^\mu{}_\nu u^\nu . \longrightarrow \ddot{x} = \frac{q}{mc} F \dot{x} \quad (1)$$

donde $F \equiv [F^\mu{}_\nu] = \begin{bmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix}$

$\vec{B} = B\hat{z}$, $\vec{E} = E\hat{x}$ $\Rightarrow$ $F = E \begin{pmatrix} 0&1&0&0 \\ 1&0&1&0 \\ 0&-1&0&0 \\ 0&0&0&0 \end{pmatrix}$

$B=E$.

La solución de (1) es:

$$\dot{x}(\tau) = e^{\frac{q}{mc}F\tau}\,\dot{x}(0).$$

Expandiendo la exponencial:

$$\dot{x}(\tau) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{q\tau}{mc}F\right)^n \dot{x}(0).$$

- $F^2 = E^2 \begin{pmatrix} 1&0&1&0 \\ 0&0&0&0 \\ -1&0&-1&0 \\ 0&0&0&0 \end{pmatrix}$, • $F^3 = \bar{\bar{0}}$

$$\Rightarrow \dot{x}(\tau) = \left[\mathbb{1} + F\frac{q\tau}{mc} + \frac{F^2}{2}\left(\frac{q\tau}{mc}\right)^2 \right] \dot{x}(0).$$

Integrando,

$$X(\tau) = \left[\mathbb{1}\,\tau + F \frac{q}{mc} \frac{\tau^2}{2} + F^2 \left(\frac{q}{mc}\right)^2 \frac{\tau^3}{6} \right] \dot{X}(0) + X(0).$$

En notación covariante:

$$\boxed{x^\mu(\tau) = u^\nu(0) \left[\underbrace{\delta^\mu_{\ \nu}\, \tau}_{①} + \underbrace{\frac{q}{mc} \frac{\tau^2}{2} F^\mu_{\ \nu} + \left(\frac{q}{mc}\right)^2}_{②} \underbrace{F^\mu_{\ \alpha} F^\alpha_{\ \nu}}_{③} \frac{\tau^3}{6} \right] + x^\mu(0)}$$

Pongamos alguna CI: $x^\mu(0) = 0$, $u^\mu(0) = (c, \vec{0})$.

① $\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} c \\ 0 \\ 0 \\ 0 \end{pmatrix} = (c, \vec{0})$

② $E \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c \\ 0 \\ 0 \\ 0 \end{pmatrix} = Ec\,(0, 1, 0, 0).$

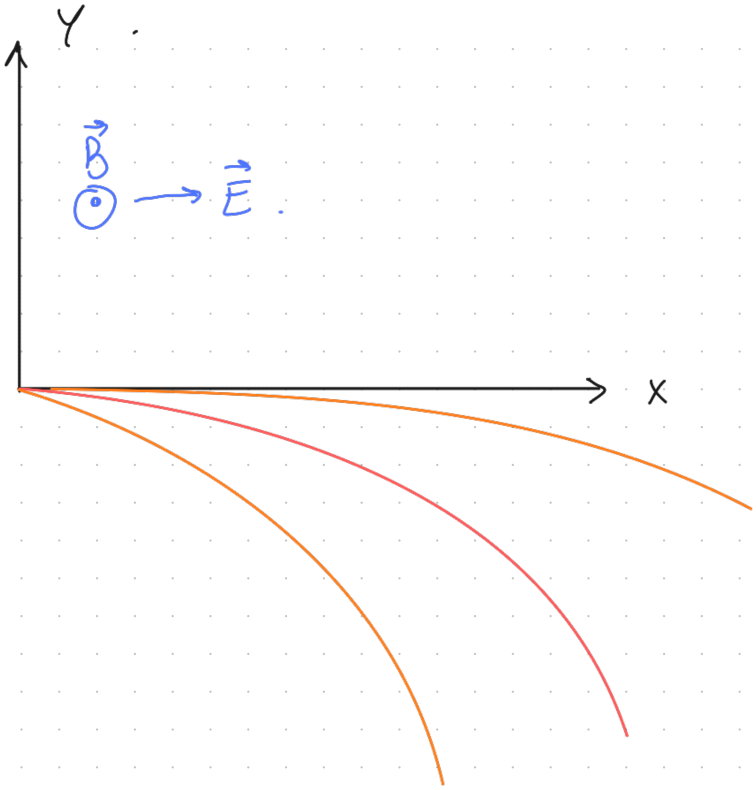
③ $E^2 \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c \\ 0 \\ 0 \\ 0 \end{pmatrix} = E^2 c\,(1, 0, -1, 0)$

$$x^\mu(\tau) = c\left[(1, \vec{0})\,\tau + \underbrace{\frac{qE}{mc} (0, \overset{\hat{x}}{1}, 0, 0) \frac{\tau^2}{2}}_{\vec{E}.} + \underbrace{\frac{q^2 E^2}{m^2 c^2} (1, 0, \overset{\hat{y}.}{-1}, 0) \frac{\tau^3}{6}}_{\vec{B}} \right]$$

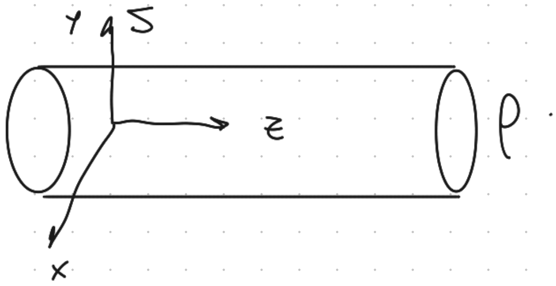
$B^2.$

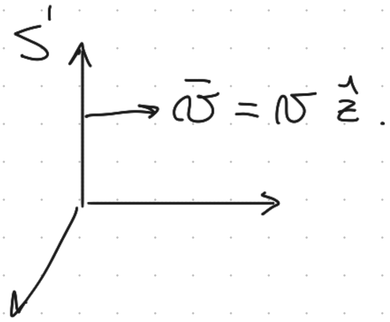
$$\begin{cases} X = \dfrac{qE}{m} \dfrac{\tau^2}{2} \longrightarrow \tau = \sqrt{\dfrac{2X}{\alpha}}, \quad \alpha = \dfrac{qE}{m}. \\[2ex] Y = -\dfrac{q^2 E^2}{m^2 c} \dfrac{\tau^3}{6} \\[2ex] ct = c\tau + \dfrac{\tau^3}{6} \cdot \dfrac{q^2 E^2}{m c^2}. \end{cases}$$

$$Y = -\frac{\alpha^2}{6c} \left(\frac{2X}{\alpha}\right)^{3/2} = -(2X)^{3/2} \frac{\sqrt{\alpha}}{6}.$$


Y
$\vec{B}$
$\vec{E}$
X

Problema 7.





$$\vec{E}(\bar{r}) = \begin{cases} 2\pi\rho r & \hat{r} \quad r \leq a \\ 2\pi\rho \dfrac{a^2}{r} & \hat{r} \quad r \geq a. \end{cases}$$

$$\vec{E}_{\parallel}' = \vec{E}_{\parallel}$$

$$\vec{E}_{\perp}' = \gamma\left(\vec{E}_{\perp} + \cancel{\vec{\beta} \times \vec{B}}\right).$$

$$\begin{cases} E_z' = E_z = 0 \\ E_\varphi' = \gamma E_\varphi = 0 \\ E_r' = \gamma E_r. \end{cases}$$

$$\Rightarrow \vec{E}'(\bar{r}) = \begin{cases} 2\pi\gamma\rho r & \hat{r} \quad r \leq a \\ 2\pi\gamma\rho \dfrac{a^2}{r} & \hat{r}. \quad r \geq a. \end{cases}$$

$$z' = \gamma(z - ct)$$

$$\left.\begin{array}{l} x' = x \\ y' = y \end{array}\right\} \hat{r},\, r. \rightarrow r = r'$$

$$\vec{E}'(\bar{r}') = \begin{cases} 2\pi\gamma\rho r' & \hat{r}' \quad r \leq a \\ 2\pi\gamma\rho \dfrac{a^2}{r'} & \hat{r}' \quad r \geq a. \end{cases}$$

Qué pasa con $\vec{B}'$?

$$\vec{B}_{\parallel}' = \vec{B}_{\parallel}$$

$$\vec{B}_{\perp}' = \gamma\left(\cancel{\vec{B}_{\perp}} - \overbrace{\vec{\beta}}^{\hat{z}} \times \overbrace{\vec{E}}^{\hat{r}}\right)$$

$$\begin{cases} B_z' = B_z = 0 \\ B_\varphi' = -\gamma\beta E_r \\ B_r' = 0 \end{cases}$$

$$\Rightarrow \vec{B}'(\vec{r}') = \begin{cases} -\frac{2\pi}{c} \overbrace{\underbrace{\gamma\beta c\rho}_{J}}^{v} r' \, \hat{\varphi} & r' < a \\ -\frac{2\pi}{c} \gamma\beta c\rho \frac{a^2}{r'} \hat{\varphi} & r' \geq a \end{cases}$$

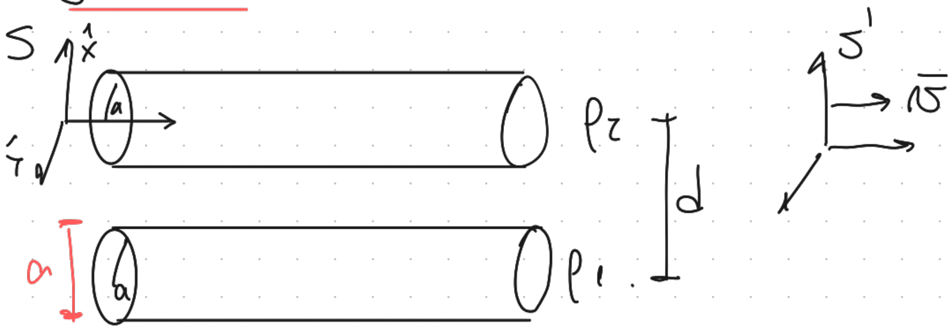
Otra forma.

$$J^\mu = (c\rho, \vec{J}) = (c\rho, \vec{0})$$

$$J'^\mu = \Lambda^\mu{}_\nu J^\nu = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} c\rho \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma\rho \\ 0 \\ 0 \\ -\gamma c\beta\rho \end{pmatrix}$$

$$J'^\mu = \gamma c\rho(1, 0, 0, -\beta).$$

Item b.



QVQ: $\frac{F}{L} = \frac{F'}{L'}$

$$\vec{F}_{12} = \int_{-\frac{L}{2}}^{L/2} \oint \rho_1 E_2 \, dl \, dS.$$

• Sup. $d \gg a$. $\Rightarrow E_2 \simeq$ cte en cil 1.

$$\vec{E}_2 \simeq -2\pi\rho_2 \frac{a^2}{d} \hat{x}.$$

$$\Rightarrow \vec{F}_1 = L \oint \rho_1 \left(-2\pi \rho_2 \frac{a^2}{d}\right) \hat{x} \, dS = \frac{-2L(\pi a^2)^2 \rho_1\rho_2 \, \hat{x}}{d}.$$

$\rho_1 = \rho_2 \equiv \rho$.

$$\Rightarrow \frac{\vec{F}}{L} = -2\frac{(\pi a^2 \rho)^2}{d} \hat{x}$$

En S' tenemos $\vec{E}'$ y $\vec{B}'$

$$\Rightarrow \vec{F}_e' = \frac{-2(\pi\gamma\rho a^2)^2}{d} L \hat{x}.$$

$$\vec{F}_m' = \frac{1}{c}\int dV\, \vec{J}_1 \times \vec{B}_2(\bar{r}_1') \qquad , \text{con } \vec{J}_1 = -c\gamma\beta\rho\,\hat{z}.$$

$$\vec{B}_2(\bar{r}_1') \approx \frac{2\pi}{c}\gamma\beta c\rho \frac{a^2}{d}\hat{y}.$$

$$\vec{F}_m' = \frac{A'}{c} L' \cdot \left(-c\gamma\beta\rho \frac{2\pi}{c}\gamma\beta c\rho\frac{a^2}{d}\right)\hat{x}.$$

$$\Rightarrow \vec{F}_{12}' = -\frac{2}{d}(\pi a^2 \rho\cancel{\gamma})^2 \underbrace{\cancel{(1-\beta)^2}}_{\gamma^{-2}} L'\hat{x}$$

$$\Rightarrow \frac{\vec{F}_{12}'}{L'} = -\frac{2}{d}(\pi a^2\rho)^2 \cdot \hat{x}.$$

Generalización de la fuerza de Lorentz.

$$f^\mu = \frac{1}{c} F^\mu{}_\nu j^\nu. \longrightarrow f^\mu = \frac{1}{c}(\underbrace{\vec{J}\cdot\vec{E}}_{\text{Joule.}}, \underbrace{c\rho\vec{E} + \vec{J}\times\vec{B}}_{F_{\text{Lorentz}}}).$$

- En S:

$$f^\mu = \left(0, \frac{2\pi\rho^2 a^2}{d}, 0, 0\right)$$

$$f^\mu f_\mu =$$

- En S':

$$f'^\mu = \Lambda^\mu{}_\nu f^\nu = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} 0 \\ \frac{2\pi\rho^2 a^2}{d} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{2\pi\rho^2 a^2}{d} \\ 0 \\ 0 \end{pmatrix}$$

$$F^{\mu}{}_{\nu} \equiv F \qquad \mathcal{F}^{\mu}{}_{\nu} \equiv G$$

$$F^0 = \mathbb{1}$$

$$F^2 = G^2 + \mathbb{1}\, k_1$$

$$FG = GF = \frac{1}{2}\mathbb{1}\, k_2$$

$$k_1 = |\vec{E}|^2 - |\vec{B}|^2$$

$$k_2 = 2\,(\vec{E}\cdot\vec{B})$$

$$F^3 = \frac{1}{2} k_2\, G + k_1 F.$$

$$X \sim \underbrace{\underbrace{f_1}_{\cos\tau}\, \mathbb{1} + \underbrace{f_2}_{\sin\tau}\, F}_{B} + \underbrace{\underbrace{f_3}_{\cosh\tau}\, F^2 + \underbrace{f_4}_{\sinh\tau}\, G}_{E}.$$