Journal of Dynamics and Differential Equations, Vol. 3, No. 1, 1991

The Twisting Tennis Racket

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Received July 19, 1989

This paper describes, analyzes, and explains a novel twisting phenomenon which occurs in a triaxial rigid body (such as a tennis racket) when it is rotating about an axis initially near its unstable intermediate principal axis.

KEY WORDS: Euler equations; rigid body motion; Euler angles; Eulerian wobble.
1980 AMS(MOS) MATHEMATICS SUBJECT CLASSIFICATIONS: 70E15, 58F05.

1. INTRODUCTION

The classical treatments of the dynamics of a tennis racket about its intermediate axis fail to describe a remarkable aspect of its motion which is revealed in the following experiment. Mark the faces of the racket so that they can be distinguished. Call one rough and the other smooth. Hold the racket horizontally by its handle with the smooth face up. Toss the racket into the air attempting to make it rotate about the intermediate axis (namely, the axis in the plane of the face which is perpendicular to the handle). After one rotation, catch the racket by the handle. The rough face will almost always be up! In other words, the racket typically makes a half-twist about its handle.

The experiment above was shown to one of us (R.C.) by Professor W. Burke of the University of California at Santa Cruz. The twisting phenomenon seems to be new. It is not mentioned in a recent article on the Eulerian wobble (Colley, 1987), in general texts on classical mechanics (Arnol'd, 1978; Goldstein, 1950; Landau and Lifschitz, 1976), or in

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specialized texts on rigid body motion (Klein and Sommerfeld, 1897–1910; Webster, 1920).

In this paper we explain the twist by analyzing the equations of motion of the tennis racket in *space*. These differential equations, which we call the *full Euler equations* (see Section 2), are given in terms of suitably chosen Euler angles. Our treatment of the twist has two main parts. In the first part we prove two theorems which show that the handle moves nearly in a plane and rotates nearly uniformly (see Section 3). The near-planarity of the motion of the handle allows us to define what we mean when we say that the racket undergoes a "half-twist" about its handle. Namely, let $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ be a frame which corotates with the racket and lies along its principal axes (see Fig. 1). We say that the racket performs a half-twist about its handle (which lies along $\hat{e}_1$), if the vector $\hat{e}_2$ (which lies along the intermediate axis of the racket) crosses from being perpendicular to and lying on one side of the plane of motion of the handle to being perpendicular to and lying on the other side of the plane of motion of the handle. The fact that the handle rotates nearly uniformly means that when we stop the experiment, the amount of twist will be reproducible. In the second part, we discuss how the twist and rotation of the handle are related. More specifically, we demonstrate that for a high percentage of initial conditions which lead to near-rotations about the intermediate axis, the racket will perform a near-half-twist in the time it takes the handle to make a full rotation. This we do by analyzing a special case where the motion is along the unstable manifold (see Section 4). In the Appendix we prove an estimate which gives the size of a region of initial conditions where a near-half-twist does not take place. These results are then combined with numerical studies which show that, for most suitable initial conditions, the handle does perform a near-half-twist.

2. THE FULL EULER EQUATIONS

The problem, of course, is to explain the twist as a consequence of the classical mechanics of a rotating tennis racket. Since the only effect of a uniform gravitational field is to cause a uniform acceleration of the center of mass of the racket, we can ignore this force in what follows. We begin by discussing Euler's equations for the components of the angular momentum $\mathbf{M} = (M_1, M_2, M_3)$ of the tennis racket in a noninertial frame $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ which corotates with the racket and lies along its principal axes (Fig. 1). We then make a choice of Euler angles and derive the full Euler equations which describe the evolution of the orientation of the racket in space.

The components of the angular momentum of a tennis racket about its

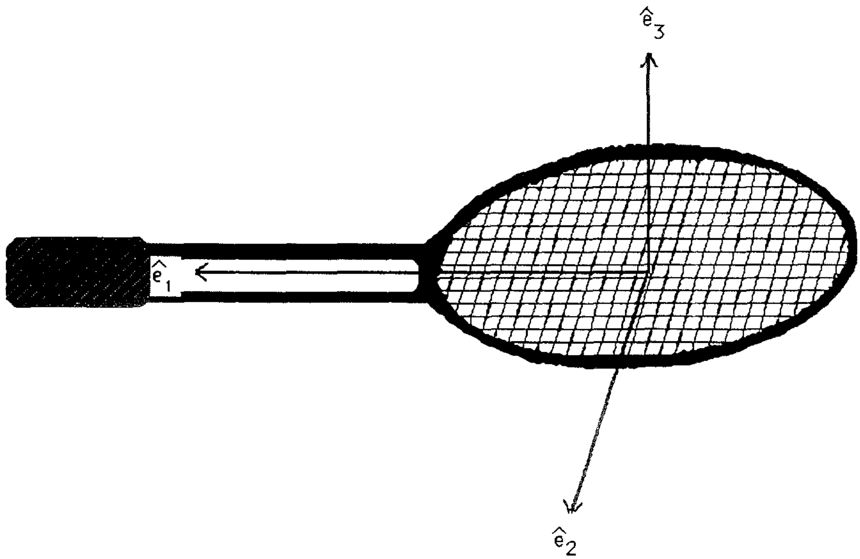

Fig. 1. The principal axes of a tennis racket.

center of mass are governed by Euler's equations (Goldstein, 1950; Landau and Lifschitz, 1976):

$$\begin{aligned} \frac{dM_1}{dt} &= -\left(\frac{1}{I_2} - \frac{1}{I_3}\right) M_2 M_3 \\ \frac{dM_2}{dt} &= \left(\frac{1}{I_1} - \frac{1}{I_3}\right) M_1 M_3 \\ \frac{dM_3}{dt} &= -\left(\frac{1}{I_1} - \frac{1}{I_2}\right) M_1 M_2 \end{aligned} \tag{1}$$

Here I_1, I_2, I_3 are the principal moments of inertia of the racket, which we assume satisfy

$$0 < I_1 < I_2 < I_3 \tag{2}$$

$$I_1 \ll I_2 \tag{3}$$

and

$$I_1 + I_2 \approx I_3\text{[4]} \tag{4}$$

[4] For a standard tennis racket such as the Wilson T-2000 the values of I_1, I_2, and I_3 are $I_1 = 0.00121$ kg-m^2, $I_2 = 0.01638$ kg-m^2, and $I_3 = 0.01748$ kg-m^2 as measured by Brody (1985). These values are used in all of our numerical examples.

Qualitatively the solutions of (1) lie on the intersection of the energy ellipsoid

$$E=\frac{1}{2}\left(\frac{M_1^2}{I_1}+\frac{M_2^2}{I_2}+\frac{M_3^2}{I_3}\right) \tag{5}$$

and the angular momentum sphere

$$M^2=M_1^2+M_2^2+M_3^2 \tag{6}$$

because E and M^2 are conserved quantities (see Fig. 2).

Quantitatively the solutions of Euler's equations are given by Jacobi elliptic functions (Abramowitz and Stegun, 1964; Gradshteyn and Ryzhik, 1980; Tricomi, 1953; Rauch and Lebowitz, 1973). The nonequilibrium solutions of (1) break up into three cases depending on whether

$$\delta=M^2-2I_2E \tag{7}$$

is positive, negative, or zero. These solutions are displayed in Table I. In the table,

$$A_1=\sqrt{\frac{I_1(2I_3E-M^2)}{I_3-I_1}}, \qquad A_1'=M\sqrt{\frac{I_1(I_3-I_2)}{I_2(I_3-I_1)}},$$

$$\tau_1=t\sqrt{\frac{(I_3-I_2)(M^2-2I_1E)}{I_1I_2I_3}}+c=Bt+c$$

$$A_2=\sqrt{\frac{I_2(2I_3E-M^2)}{I_3-I_2}}, \qquad A_2'=\sqrt{\frac{I_2(M^2-2I_1E)}{I_2-I_1}},$$

$$\tau_2=\frac{tM}{I_2}\sqrt{\frac{(I_3-I_2)(I_2-I_1)}{I_1I_3}}+c=B_0t+c$$

$$A_3=\sqrt{\frac{I_3(M^2-2I_1E)}{I_3-I_1}}, \qquad A_3'=M\sqrt{\frac{I_3(I_2-I_1)}{I_2(I_3-I_1)}},$$

$$\tau_3=t\sqrt{\frac{(I_2-I_1)(2I_3E-M^2)}{I_1I_2I_3}}+c=B't+c$$

Table I. Solutions of Euler's Equations

	$\delta>0$	$\delta=0$	$\delta<0$
M_1	$\varepsilon A_1\,\mathrm{cn}(\tau_1, m)$	$\varepsilon_1 A_1'\,\mathrm{sech}\,\tau_2$	$\varepsilon A_1\,\mathrm{dn}(\tau_3, m')$
M_2	$A_2\,\mathrm{sn}(\tau_1, m)$	$\varepsilon_2 M\tanh\tau_2$	$A_2'\,\mathrm{sn}(\tau_3, m')$
M_3	$\varepsilon A_3\,\mathrm{dn}(\tau_1, m)$	$\varepsilon_1\varepsilon_2 A_3'\,\mathrm{sech}\,\tau_2$	$\varepsilon A_3\,\mathrm{cn}(\tau_3, m')$

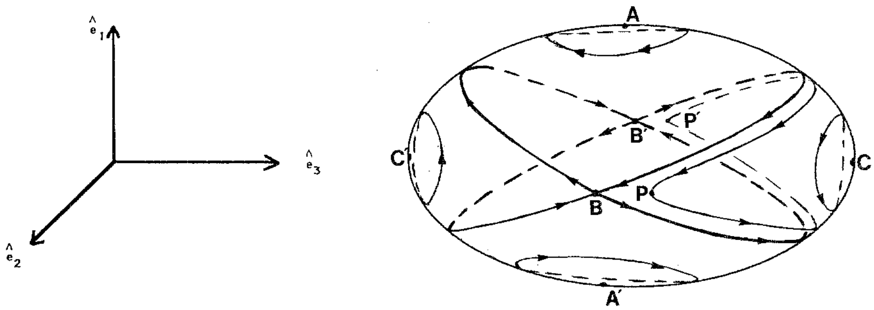

Fig. 2. Solutions of Euler's equations for a fixed value of energy and angular momentum.

and

$$m = k^2 = \frac{(I_2 - I_1)(2I_3E - M^2)}{(I_3 - I_2)(M^2 - 2I_1E)} = \frac{1}{m'}$$

The constants c, ε, ε_1, ε_2 are determined by the initial conditions. When $\delta \neq 0$ and E, M are fixed, the intersection of the energy ellipsoid (5) with the angular momentum sphere (6) is *two* closed orbits of (1). The choice of ε is determined as follows. If $\delta > 0$, then

$$\varepsilon = \begin{cases} 1, & \text{if } M_3 > 0 \\ -1, & \text{if } M_3 < 0 \end{cases}$$

If $\delta < 0$, then

$$\varepsilon = \begin{cases} 1, & \text{if } M_1 > 0 \\ -1, & \text{if } M_1 < 0 \end{cases}$$

When $\delta = 0$, there are four trajectories of (1) which lie on the intersection of the energy ellipsoid (5) with the planes

$$\Pi_{\pm}: \sqrt{I_3(I_2 - I_1)}\, M_1 = \pm\sqrt{I_1(I_3 - I_2)}\, M_3 \tag{8}$$

They are the stable and unstable manifolds of the hyperbolic equilibrium points B: $(0, \sqrt{2I_2E}, 0)$ and B': $(0, -\sqrt{2I_2E}, 0)$ (see Fig. 2). The choice of $\varepsilon_2 = \pm 1$ determines whether we are on the stable or unstable manifold of B, while the choice of ε_1 determines which of the two branches of these manifolds we are on.

The Eulerian picture of the motion of the tennis racket has two short-comings. The first one is that it is based on the noninertial frame $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ which is corotating with the body. To remedy this, we recall that **M** is a constant of the motion of the tennis racket when referred to an inertial frame (X, Y, Z) *fixed* in space. Thus, apparent motions of **M** along trajectories of (1) in the noninertial frame translate into rotations of the triad $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ with respect to $\mathbf{M}$ in the fixed spatial frame. In particular, we can determine the angles between $\mathbf{M}$ and each $\hat{e}_i$ from this picture. However, we *cannot* determine the amount of rotation about $\mathbf{M}$. This second defect can be corrected only by going to the full Euler equations which describe the motion of the racket in the spatial frame in terms of Euler angles.

In the remainder of this paper we use the two choices of Euler angles which are illustrated in Fig. 3. In Euler angles I the vector $\mathbf{M}$, which lies along the Z-axis, has components

$$M_1 = M \sin\theta \cos\psi, \quad M_2 = M \cos\theta, \quad M_3 = M \sin\theta \sin\psi \tag{9}$$

in Euler angles II it has components

$$M_1 = M \cos\theta, \quad M_2 = M \sin\theta \sin\psi, \quad M_3 = M \sin\theta \cos\psi \tag{10}$$

The full Euler equations for Euler angles I are obtained by first solving the equations

$$\frac{M_1}{I_1} = \dot{\phi} \sin\theta \cos\psi - \dot{\theta} \sin\psi$$

$$\frac{M_2}{I_2} = \dot{\phi} \cos\theta + \dot{\psi}$$

$$\frac{M_3}{I_3} = \dot{\phi} \sin\theta \sin\psi + \dot{\theta} \cos\psi$$

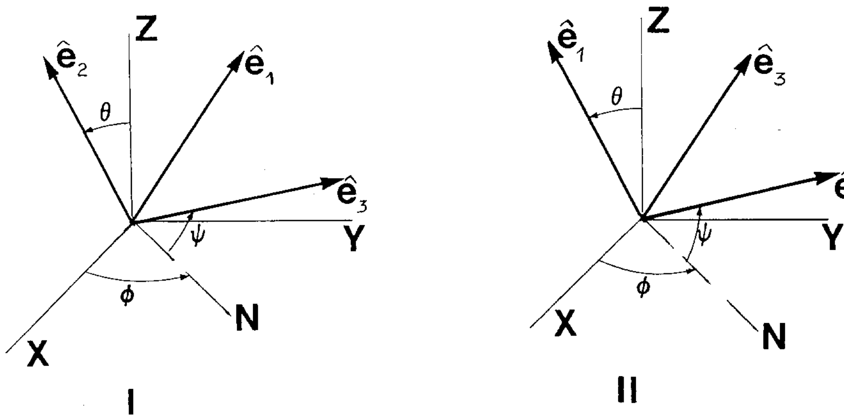

Fig. 3. Two choices of Euler angles. N is the line of nodes: the intersection of the $\hat{e}_1$–$\hat{e}_3$ plane in I (the $\hat{e}_2$–$\hat{e}_3$ in II) with the X–Y plane

for the angular velocities $\dot{\theta}$, $\dot{\psi}$, $\dot{\phi}$ about N, $\hat{e}_2$, and the Z-axis, respectively, and then using (9). This gives

$$\begin{aligned}
\frac{d\theta}{dt} &= -M\left(\frac{1}{I_1}-\frac{1}{I_3}\right)\sin\theta\sin\psi\cos\psi \\
\frac{d\phi}{dt} &= \frac{M}{I_3}\sin^2\psi+\frac{M}{I_1}\cos^2\psi \\
\frac{d\psi}{dt} &= \left(\frac{M}{I_2}-\frac{M\sin^2\psi}{I_3}-\frac{M\cos^2\psi}{I_1}\right)\cos\theta
\end{aligned} \tag{11}$$

A similar argument shows that the full Euler equations for Euler angles II are

$$\begin{aligned}
\frac{d\theta}{dt} &= M\left(\frac{1}{I_2}-\frac{1}{I_3}\right)\sin\theta\sin\psi\cos\psi \\
\frac{d\phi}{dt} &= \frac{M}{I_2}\sin^2\psi+\frac{M}{I_3}\cos^2\psi \\
\frac{d\psi}{dt} &= \left(\frac{M}{I_1}-\frac{M\sin^2\psi}{I_2}-\frac{M\cos^2\psi}{I_3}\right)\cos\theta
\end{aligned} \tag{12}$$

[see Goldstein (1950) and Landau and Lifschitz (1976) for more details].

3. THE MOTION OF THE HANDLE IN SPACE

In this section we prove two theorems which show that the handle of the tennis racket rotates nearly uniformly and nearly in a plane.

First we show that the handle moves nearly in a plane. Let

$$t \to (\hat{e}_1(t), \hat{e}_2(t), \hat{e}_3(t))$$

be the time evolution of the frame in Fig. 3 governed by the full Euler equations (12). Thinking of $\hat{e}_1(t)$ as a point moving on the unit sphere, we show that it stays in a small band about the equator of the sphere, which lies on the X–Y plane. Let $\alpha(t)$ be the angle between $\hat{e}_1(t)$ and the X–Y plane. We prove the following.

Theorem 1. *Given E and M, then for all $t \in \mathbb{R}$*

$$0 \leqslant \tan\alpha(t) \leqslant \sqrt{\frac{I_1(2I_3E-M^2)}{I_3(M^2-2I_1E)}} \tag{13}$$

Proof. In Euler angles *II*, it follows that

$$\sin\alpha(t) = |\cos\theta(t)| = \frac{|M_1(t)|}{M}$$

since $\alpha(t) = |\pi/2 - \theta(t)|$. Therefore we need only look for the maximum value of $|M_1(t)|$. The argument now breaks up into three cases depending on whether δ is positive, negative, or zero.

Suppose that $\delta > 0$. Then from the second column in Table I and the fact that $|\mathrm{cn}(t, m)| \leqslant 1$, we find that the maximum value of $\sin\alpha(t)$ is

$$\frac{A_1}{M} = \frac{1}{M}\sqrt{\frac{I_1(2I_3E - M^2)}{I_3 - I_1}} \tag{14}$$

which, after a bit of manipulation, gives (13).

Suppose that $\delta < 0$. Then from the fourth column in Table I and the fact that $1 - m \leqslant \mathrm{dn}(t, m) \leqslant 1$, we find that the maximum of $\sin\alpha(t)$ is again given by (14).

Suppose that $\delta = 0$. Then from the third column in Table I and the fact that $\mathrm{sech}(t) \leqslant 1$, we find that the maximum of $\sin\alpha(t)$ is given by

$$\frac{A_1'}{M} = \sqrt{\frac{I_1(I_3 - I_2)}{I_2(I_3 - I_1)}}$$

which agrees with (14), because $M^2 = 2I_2E$ in this case. ▮

If we use the approximations

$$M^2 = 2I_2E \qquad \text{and} \qquad I_3 \approx I_1 + I_2$$

then the right-hand side of (13) becomes

$$\sqrt{\frac{I_1^2}{I_2^2 - I_1^2}} \approx \frac{I_1}{I_2}$$

using (3). Therefore $\alpha(t)$ is small for a tennis racket whose principal moments of inertia satisfy (2), (3), and (4).[5]

Let $\beta(t)$ be the angle that the vector $\hat{e}_2$ makes with the Z-axis (in Euler angles *I* this is θ). Then the amount of twist that the tennis racket makes about its handle in time t is

$$\tau(t) = \beta(t) - \beta(0) \approx \beta(t)$$

[5] For the standard tennis racket, substituting $M^2 = 2I_2E$ and the values of I_1, I_2, I_3 into (13), we find that $\alpha(t)$ is at most 4.05°.

The approximation is valid because the face of the racket is initially nearly horizontal, $\beta(0) \approx 0$. The following theorem shows that the handle rotates nearly uniformly. Therefore, the amount of twist $\tau(T)$ after one revolution is a quantity which can be reproducibly measured. More precisely, we show that the projection of $\hat{e}_1(t)$ on the X–Y plane moves almost uniformly. Using Euler angles *II*, this means that $\phi(t)$ (which measures the rotation of the handle) increases nearly linearly with time. We now prove the following.

Theorem 2. *Let $\theta(t)$, $\phi(t)$, $\psi(t)$ be solutions of the full Euler equations (12). Then for all $t \geqslant 0$,*

$$\frac{M}{I_3} t \leqslant \phi(t) - \phi_0 \leqslant \frac{M}{I_2} t \tag{15}$$

Moreover, there are constants J_0, J_1, $\delta_1 \geqslant 0$, $\delta_2 \geqslant 0$ such that

$$-\delta_1 \leqslant \phi(t) - (J_0 t + J_1) \leqslant \delta_2 \tag{16}$$

and

$$\frac{M}{I_3} \leqslant J_0 \leqslant \frac{M}{I_2} \tag{17}$$

Proof. Since $I_2 < I_3$, (15) follows immediately from the second equation of (12).

To prove (16), we must use the explicit solutions of Euler's equations given in Table I. We argue case by case based on the sign of δ. When $\delta > 0$, after some straightforward manipulations using (10), the second equation in (12) reads

$$\frac{d\phi}{dt} = M \frac{(M^2 - 2I_1 E) + (2I_3 E - M^2)\,\mathrm{sn}^2(Bt + c, m)}{I_3(M^2 - 2I_1 E) + I_1(2I_3 E - M^2)\,\mathrm{sn}^2(Bt + c, m)} \tag{18}$$

Since $\mathrm{sn}^2(t, m)$ is period $2K$, where $K = K(m)$ is the complete elliptic integral

$$\int_0^1 [(1 - t^2)(1 - mt^2)]^{-1/2}\, dt$$

from (18) we see that $d\phi/dt$ is periodic of period $2K/B$. Let J_0 be the average of the right-hand side of (18) over a period, then a straightforward change of the variable of integration gives

$$J_0 = \frac{M}{K} \int_0^K \frac{(M^2 - 2I_1 E) + (2I_3 E - M^2)\,\mathrm{sn}^2(s, m)}{I_3(M^2 - 2I_1 E) + I_1(2I_3 E - M^2)\,\mathrm{sn}^2(s, m)}\, ds$$

Let $\phi_1(t) = \phi(t) - J_0 t$. Then ϕ_1 is periodic of period $2K/B$, since

$$\begin{aligned}\phi_1\left(t+\frac{2K}{B}\right) &= \phi\left(t+\frac{2K}{B}\right) - J_0\left(t+\frac{2K}{B}\right)\\ &= \phi(t) + \int_0^{2K/B} \frac{d\phi}{ds}(t+s)\,ds - J_0\left(t+\frac{2K}{B}\right)\\ &= \phi(t) + J_0\frac{2K}{B} - J_0 t - J_0\frac{2K}{B}\\ &= \phi_1(t)\end{aligned}$$

Let J_1 be the average of ϕ_1 over a period and put $\phi_2(t) = \phi_1(t) - J_1$. Then ϕ_2 is a continuous periodic function with average 0. Hence there are constants $\delta_1, \delta_2 \geqslant 0$ such that $-\delta_1 \leqslant \phi_2(t) \leqslant \delta_2$ for all t. This establishes (16). When $\delta < 0$, the argument can be carried out in essentially the same manner. When $\delta = 0$, we have

$$\frac{d\phi}{dt} = M\frac{(I_2 - I_1) + (I_3 - I_2)\tanh^2(Bt+c)}{I_3(I_2 - I_1) + I_1(I_3 - I_2)\tanh^2(Bt+c)}$$

which can be explicitly integrated to give

$$\phi(t) = C + \frac{M}{I_2}t - \tan^{-1}\left[\sqrt{\frac{I_1(I_3 - I_2)}{I_3(I_2 - I_1)}}\tanh(Bt+c)\right]$$

where C is a constant of integration. Putting $J_0 = M/I_2$ and $J_1 = C$ gives (16) with

$$\delta_1 = \delta_2 = \tan^{-1}\left[\sqrt{\frac{I_1(I_3 - I_1)}{I_3(I_2 - I_1)}}\right]$$

To finish the argument we need to prove (17). From the definition

$$J_0 = \frac{B}{2K}\int_0^{2K/B}\frac{d\phi}{dt}\,dt = \frac{B}{2K}\left(\phi\left(\frac{2K}{B}\right) - \phi(0)\right) \tag{19}$$

Substituting $t = 2K/B$ into (15) and then using (19) yields

$$\frac{M}{I_3}\frac{2K}{B} \leqslant \frac{2K}{B}J_0 \leqslant \frac{M}{I_2}\frac{2K}{B}$$

Multiplying through by $B/2K > 0$ gives (17). ∎

Although the theorems in this section do not prove that the tennis racket twists, they do show that the handle's motion is close to being uniform periodic rotation in the X–Y plane. This allows us to make the definition of half-twist presented in Section 1 meaningful and to formulate a simple criterion for idealizing the experiment of tossing a tennis racket, viz., the toss is completed when the handle returns close to its starting point, as compared to an ideal toss of a racket rotating uniformly in a plane. Once it is known that the handle returns close to its starting point, the twist angle is measured by the Euler angle θ (in Euler angles I).

4. AN EXPLANATION OF THE TWIST

In this section we give an explanation of the twist of the tennis racket. Informally stated our analysis shows the following.

(a) The twist occurs when **M** starts near—but not too near (see the Appendix)—the hyperbolic equilibrium point B on its unstable manifold.

(b) There is a characteristic time for the half-twist. Moreover, the time it takes for the handle to complete one revolution is larger than the characteristic twisting time.

(c) Because B and B' are hyperbolic equilibrium points, trajectories of Euler's equations traverse neighborhoods of B and B' very slowly.

These facts constitute our explanation for the twist. Here (a) states that the twist exist, (b) that the handle will have enough time to twist, and (c) that the racket is likely to be caught after a half-twist.

We begin by discussing (a). To establish that a half-twist occurs as **M** traverses the unstable manifold of B to B', we use Euler angles I. From (8) and (9) note that the stable and unstable manifolds of B are given by

$$\tan\psi = \pm\sqrt{\frac{I_3(I_2-I_1)}{I_1(I_3-I_2)}} \tag{20}$$

If our initial $\psi = \psi_0$ satisfies (20), then (11) reduces to

$$\begin{aligned}\frac{d\theta}{dt} &= -M\left(\frac{1}{I_1}-\frac{1}{I_3}\right)\sin\theta\sin\psi\cos\psi\\ \frac{d\phi}{dt} &= \frac{M}{I_2}\\ \frac{d\psi}{dt} &= 0\end{aligned} \tag{21}$$

where $M = \sqrt{2I_2E}$. Integrating (21) gives

$$\psi = \psi_0, \qquad \phi = \frac{M}{I_2}t + \phi_0, \qquad \cos\theta = \pm\tanh(B_0 t + c)$$

where c is an integration constant and B_0 is given in Table I. Note that θ is the amount of twist around the handle. Even though the time taken to go from B to B' along the unstable manifold is infinite, most of the twist takes place in a characteristic time of $2/B_0$; see Fig. 4.

In order to verify (b) we note that theorem 2 implies that the time T needed for the handle to make one revolution is at least $2\pi I_2/M$. This is larger than the characteristic twisting time, because

$$\begin{aligned}\frac{2}{B_0} &= \frac{2I_2}{M}\sqrt{\frac{I_1 I_3}{(I_3 - I_2)(I_2 - I_1)}}\\ &\approx \frac{2I_2}{M}\sqrt{\frac{I_2 + I_1}{I_2 - I_1}} \qquad \text{(using the approximation } I_3 \approx I_1 + I_2\text{)}\\ &\approx \frac{2I_2}{M} \qquad \text{(using } I_1 \ll I_2\text{)}\end{aligned}$$

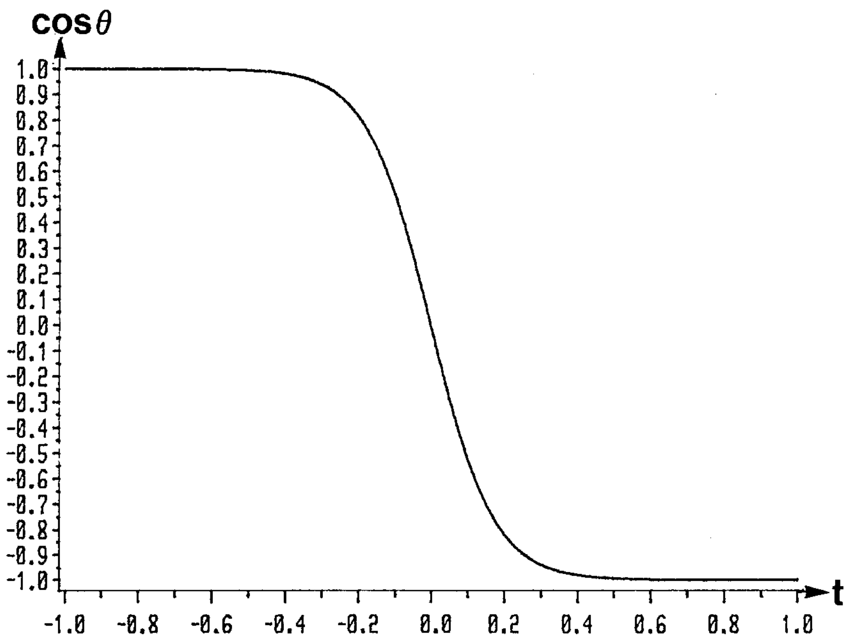

Fig. 4. The graph of $\cos\theta = -\tanh B_0 t$. Here the characteristic time for the twist is 0.35836 sec when $E = 0.32333$ J. The time for one revolution of the handle is 1 sec.

To verify (c) we linearize Euler's equations at the hyperbolic equilibrium points B and B' with coordinates $(0, \pm\sqrt{2I_2E}, 0)$ on the angular momentum sphere (5). In the tangent plane at B or B' with coordinates $(M_1, 0, M_3)$, we obtain the system of linear differential equations

$$\begin{aligned} \dot{M}_1 &= \mp a M_3 \\ \dot{M}_3 &= \mp b M_1 \end{aligned} \tag{22}$$

where

$$a = \left(\frac{1}{I_2} - \frac{1}{I_3}\right)\sqrt{2I_2E} \qquad \text{and} \qquad b = \left(\frac{1}{I_1} - \frac{1}{I_2}\right)\sqrt{2I_2E}$$

Choosing new coordinates (ξ, η) so that

$$\begin{aligned} M_1 &= \sqrt{a}\,(\xi + \eta) \\ M_3 &= \sqrt{b}\,(\xi - \eta) \end{aligned}$$

(22) becomes

$$\begin{aligned} \dot{\xi} &= \mp \sqrt{ab}\,\xi \\ \dot{\eta} &= \pm \sqrt{ab}\,\eta \end{aligned} \tag{23}$$

Now choose small segments on $\eta = d$ and $\xi = -d$, called sections, which meet the coordinate axes and are transverse to the trajectories of (23) which they intersect (see Fig. 5). The neighborhood transit time T_N for

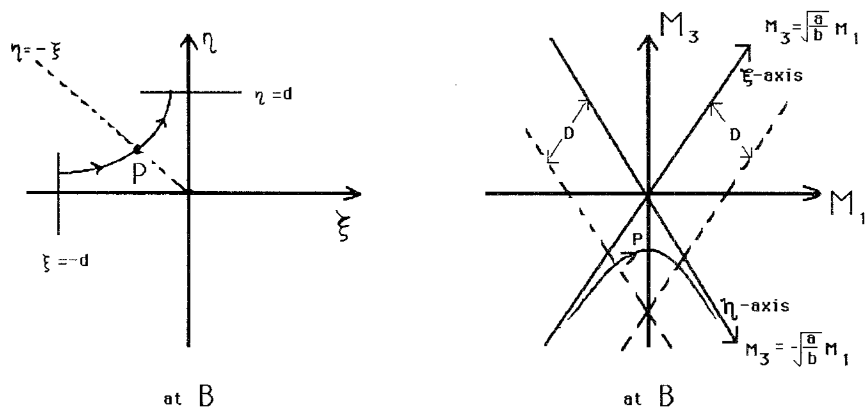

Fig. 5. A neighborhood of the hyperbolic equilibrium point B.

the trajectory of (23) through $P=(-\delta/\sqrt{2}, \delta/\sqrt{2})$ to pass from the section $\xi=-d$ to the section $\eta=d$ is easily computed to be

$$T_N=\frac{2}{\sqrt{ab}}\ln\frac{\sqrt{2}\,d}{\delta}$$

Returning to the original coordinates on the energy ellipsoid, the transit time between the corresponding transverse segments, which to first order lie at a distance $D=d\sqrt{a+b}$ along the stable and unstable manifolds, is to first approximation T_N. Thus the neighborhood transit time for B and B' is $2T_N$. The remaining part of the period of the trajectory of (1) through P is twice the time T_S needed to go from the section on the unstable manifold at B' to the symmetrically chosen section on the stable manifold of B. By continuity with respect to parameters, T_S approaches a finite limit $T_S(0)$ as δ goes to 0. In fact $T_S(0)$ represents the transit time between the sections for the trajectory along the separatrix. Thus, the ratio of the neighborhood transit time to the period of the trajectory has limiting value 1 as δ goes to 0. This makes precise the statement that a point on the trajectory spends most of its time in a neighborhood of one of the equilibrium points B or B'.

This completes our demonstration that the face of the tennis racket twists about the handle.

5. NUMERICAL RESULTS

To show for a large percentage of suitable initial conditions that the tennis racket does perform a near-half-twist, we have simulated the tennis racket experiment described in Section 1 on a computer by integrating the full Euler equations (11). This allows us to simulate performing a large number of experiments with a variety of initial conditions. We use the values of I_1, I_2, and I_3 of the standard racket and the value 0.32333 for E. We choose the initial ψ_0 from $[0, \pi)$ and take small initial values of θ_0. The total angular momentum M is then determined by

$$M^2=2E\left(\frac{\cos^2\psi_0\sin^2\theta_0}{I_1}+\frac{\cos^2\theta_0}{I_2}+\frac{\sin^2\psi_0\sin^2\theta_0}{I_3}\right)^{-1}$$

using (5) and (9). Finally, we choose ϕ_0 so as to make $\hat{e}_1$ (that is, the racket handle) as near to the direction of the positive Y-axis as possible for the given values of θ_0 and ψ_0. This is accomplished by taking ϕ_0 to be the angle between $(\cos\psi_0\cos\theta_0, -\sin\psi_0)$ and $(1, 0)$, that is,

$$\phi_0=\arctan\left(\frac{-\sin\psi_0}{\cos\psi_0\cos\theta_0}\right)$$

Before we can proceed further we need an estimate of the amount of scatter about $\theta = 0$ introduced by a person trying to toss a tennis racket about its intermediate axis. We make the following experimental observation: tosses which lead to rotation about the $\hat{e}_3$-axis are nearly always stable. This result is useful, because looking at Fig. 2 we see that significant deviations from rotation about the $\hat{e}_3$-axis lead to trajectories which stray far away from the equilibrium point. Since such large instabilities are not observed, we can be sure that, with a little practice, one can toss a racket so that its axis of rotation is within a few degrees of $\hat{e}_3$. We assume that this tolerance can be carried over to rotations about $\hat{e}_2$. To be specific, an error in the toss about $\hat{e}_3$ of magnitude exceeding

$$\tan^{-1}\left[\sqrt{\frac{I_1(I_3 - I_2)}{I_3(I_2 - I_1)}}\right] \approx 0.07073 \text{ radian} \approx 4.05^\circ$$

would lead to wide excursions. Therefore we assume that $\theta_0 \in [0, 0.025]$.

Using these initial conditions $(\theta_0, \psi_0) \in [0, 0.025] \times [0, \pi)$, we integrated (11) until the angle of the projection of $\hat{e}_1(t)$ on the X–Y plane increased by 2π. In other words, we kept track of the polar angle of the vector

$$(-\sin\psi\cos\phi - \cos\psi\cos\theta\sin\phi,\ -\sin\psi\sin\phi + \cos\psi\cos\theta\cos\phi)$$

Finally, we checked the value of $\theta(t)$ at the stopping time. If $\hat{e}_2(t)$ was within 27° of the negative Z-axis, i.e., within 15% of completing a half-twist, then we considered that a near-half-twist had occurred; otherwise it had not. The results of the numerical experiments are presented in Fig. 6.

Figure 6 suggests that, to a good approximation, the region of no near-half-twist is a strip of constant width at an angle $\psi = 1.5001$ radians and centered on the stable manifold through B. Assuming this to be correct we can derive the half-width 0.004426 for the strip from our knowledge of how the twist occurs along the unstable manifold. This compares reasonably well with the rigorous bound $\theta \leqslant 0.00019$ for the no near-half-twist region found in the Appendix. In addition, one can use the initial conditions $(\theta_0, \psi_0) = (0.0044, 0)$ to show by numerical integration that the near-half-twist occurs. This confirms Fig. 6.

Moreover, we have computed the expected percentage of the time that the twist occurs, using the initial conditions in Fig. 6, our twist criterion, and the formula

$$\text{success ratio} = \frac{\sum_{\text{successes}} \sin\theta_i(\delta\theta_i)(\delta\psi_i)}{\sum_{\text{all points}} \sin\theta_i(\delta\theta_i)(\delta\psi_i)}$$

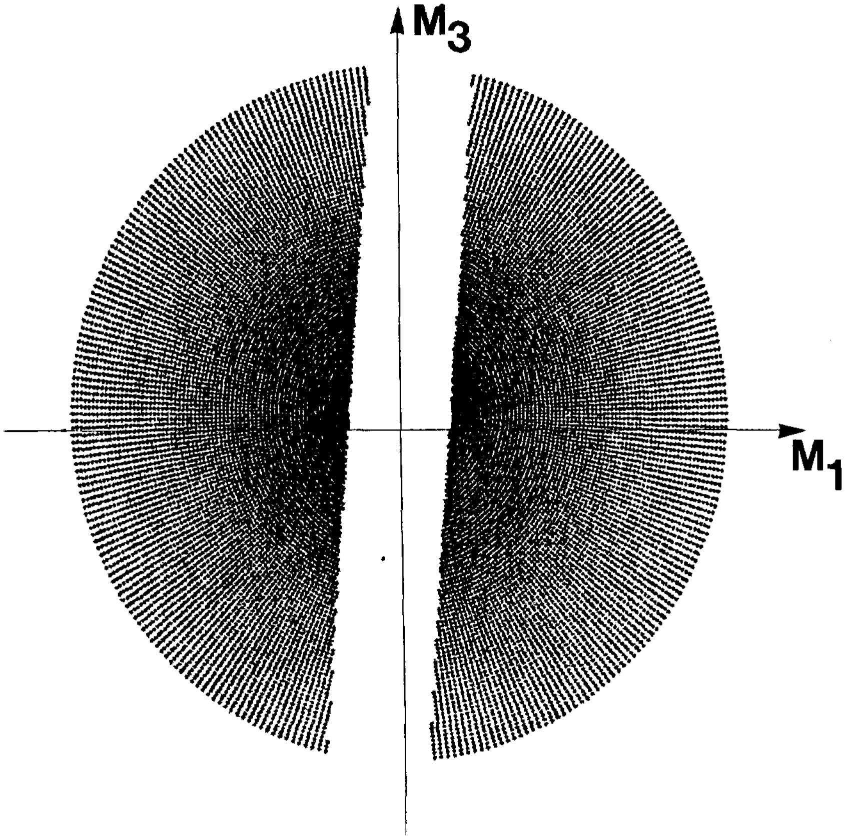

Fig. 6. The black dots are those points $(\theta \cos \psi, \theta \sin \psi)$ corresponding to initial conditions (θ, ψ) in $[0, 0.025] \times [0, 2\pi)$ where the racket makes a near-half-twist according to our criterion.

We obtained the value 0.804. Furthermore, assuming a uniform distribution of initial angular momentum vectors within the region displayed in Fig. 6, the expected amount of twist is given by the fomula

$$\text{expected amount of twist} = \frac{\sum_{\text{all points}} (\text{angle of twist}) \sin \theta_i (\delta\theta_i)(\delta\psi_i)}{\sum_{\text{all points}} \sin \theta_i (\delta\theta_i)(\delta\psi_i)}.$$

We obtained the value 2.769 radians ($\approx 159°$). This demonstrates that a twist is quite likely in our experiment.

APPENDIX

In this Appendix we give an estimate of the radius of a small ball of initial conditions about unstable equilibrium point B such that each initial condition in the ball results in the tennis racket not performing a near-half-twist. The existence of such a ball about B follows from

(a) the continuous dependence of solutions of Euler's equations on initial conditions and

(b) the fact that, if the initial condition is the unstable equilibrium, then the racket rotates uniformly about its intermediate axis and does not twist.

To estimate the region of no near-half-twist, we introduce the function

$$f(M_1, M_2, M_3) = \left(\frac{1}{I_1} - \frac{1}{I_2}\right) M_1^2 + \left(\frac{1}{I_2} - \frac{1}{I_3}\right) M_3^2 \tag{24}$$

on the energy ellipsoid

$$E = \frac{1}{2}\left(\frac{M_1^2}{I_1} + \frac{M_2^2}{I_2} + \frac{M_3^2}{I_3}\right)$$

Computing the Lie derivative of f with respect to the vector field defined by Euler's equations gives

$$\dot{f} = -4\left(\frac{1}{I_1} - \frac{1}{I_2}\right)\left(\frac{1}{I_2} - \frac{1}{I_3}\right) M_1 M_2 M_3 \tag{25}$$

Using $|M_2| \leqslant M$ and the inequality

$$|M_1 M_3| \leqslant \frac{1}{2}\left(\frac{1}{\sigma} M_1^2 + \sigma M_3^2\right)$$

where

$$\sigma = \sqrt{\frac{I_1(I_3 - I_2)}{I_3(I_2 - I_1)}}$$

we obtain the estimate

$$|\dot{f}| \leqslant 2M \sqrt{\left(\frac{1}{I_1} - \frac{1}{I_2}\right)\left(\frac{1}{I_2} - \frac{1}{I_3}\right)}\, f = 2B_0 f \tag{26}$$

Here B_0 is given in Table I. Integrating (26) gives

$$f(t) \leqslant f(0)\, e^{2B_0 t} \tag{27}$$

By (15) the largest time required for the projection of the racket handle to have completed one revolution is

$$t_0 = \frac{2\pi I_3}{M}$$

Therefore if $f(0)$ is sufficiently small, by (27) $f(t)$ cannot be too large for all $t \in [0, t_0]$.

To complete the argument we must find a bound on $f(t)$ which precludes the occurrence of a near-half-twist. Toward this goal, observe that along a solution of Euler's equations the values of $f(t) = \lambda$ determine a family of ellipses $\mathscr{E}_\lambda$ in the $M_1 - M_3$ plane. If for every $t \in [0, t_0]$, $\mathscr{E}_\lambda$ lies in the interior of the ellipse

$$\mathscr{E}\colon 2E = \frac{M_1^2}{I_1} + \frac{M_3^2}{I_3} \tag{28}$$

which is the intersection of the energy ellipsoid (5) and the plane $\{M_2 = 0\}$, then the solution $\mathbf{M}(t)$ of Euler's equations on the energy ellipsoid does not cross the $\{M_2 = 0\}$ plane. Thus no near-half-twist can occur. Since

$$I_3 - I_2 \leqslant I_2 - I_1$$

which follows from the approximations (3) and (4), we have

$$f(t) \geqslant I_3 \left(\frac{1}{I_2} - \frac{1}{I_3}\right)\left(\frac{M_1^2}{I_1} + \frac{M_3^2}{I_3}\right)$$

For $\mathscr{E}_\lambda$ to lie in the interior of $\mathscr{E}$, we need

$$f(t) < 2I_3 E\left(\frac{1}{I_2} - \frac{1}{I_3}\right)$$

Therefore if

$$f(0) < 2I_3 E\left(\frac{1}{I_2} - \frac{1}{I_3}\right) e^{-2B_0 t_0} \tag{29}$$

no near-half-twist will occur. Using Euler angles I (9) one finds that

$$\begin{aligned} f(0) &= M^2\left(\left(\frac{1}{I_1} - \frac{1}{I_2}\right)\cos^2\psi_0 + \left(\frac{1}{I_2} - \frac{1}{I_3}\right)\sin^2\psi_0\right)\sin^2\theta_0 \\ &\leqslant M^2\left(\frac{1}{I_1} - \frac{1}{I_2}\right)\sin^2\theta_0 \end{aligned} \tag{30}$$

where (θ_0, ψ_0) are the initial values of (θ, ψ). Combining (29) and (30), we see that no near-half-twist occurs if

$$M \sin \theta_0 < \sqrt{\frac{2I_1 E(I_3 - I_2)}{I_2 - I_1}} \exp\left[-2\pi \frac{I_3}{I_2} \sqrt{\frac{(I_3 - I_2)(I_2 - I_1)}{I_1 I_3}} \right] \tag{31}$$

For the standard tennis racket with $E = 0.32333$ J and $M^2 = 2I_2 E$, (25) gives

$$\theta_0 \leqslant 0.000190 \text{ radian}$$

This bound is rigorous when $M^2 < 2I_2 E$. This covers 95% of all initial conditions due to the small size of the acute angle sectors at B in Fig. 2. Even when $M^2 > 2I_2 E$ the corrections to this estimate will be very small.

ACKNOWLEDGMENTS

We are grateful to Howard Brody of the University of Pennsylvania, Enoch Durban of Princeton University, and Doug Wisbey of Wilson Sporting Goods for sharing their knowledge of the moments of inertia of various tennis rackets with us. We would also like to thank William Burke of the University of California at Santa Cruz and Tom Kane of Stanford University for their helpful criticism of the first draft of this paper.

Our computer experiments were carried out on an IBM 4381-R14 using the IMSL routine DGEAR.

This research was supported by The Air Force Office of Scientific Research under Grant AF-AFOSR-89-0078 (C.C.C.).

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