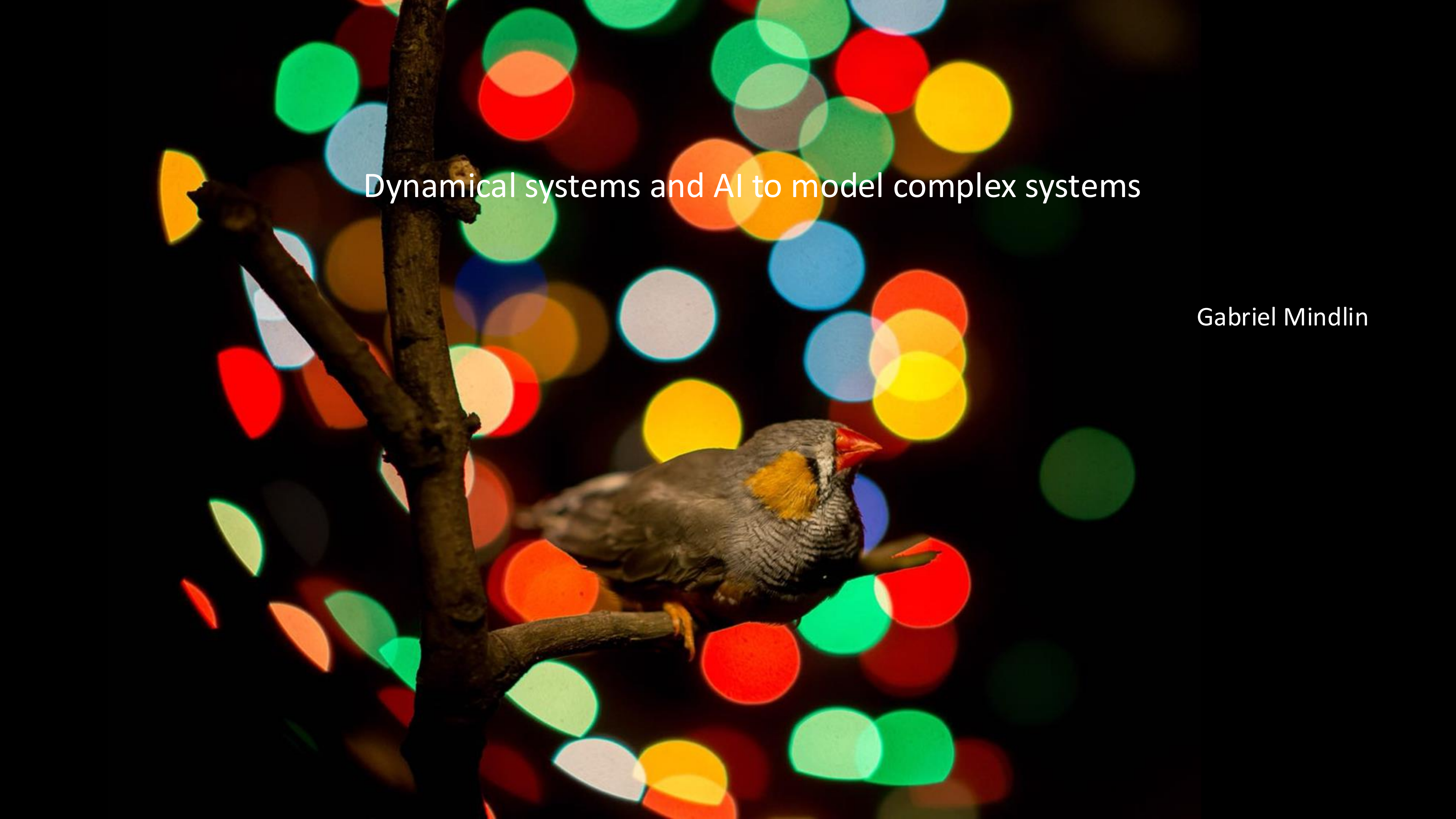
Dynamical systems and AI to model complex systems
Gabriel Mindlin

The rationale for the course:

One of the crucial intellectual decisions facing young scientists today is determining the balance between **interpretable** and **data-driven** approaches when addressing a scientific problem.

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Interpretable science

Variables have meaning

The relationship between the Variables are interpretable mechanisms

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Data driven science

A complex ”non interpretable” model (as a neural network) is trained with examples, and the model does not share with us the rationale behind its success

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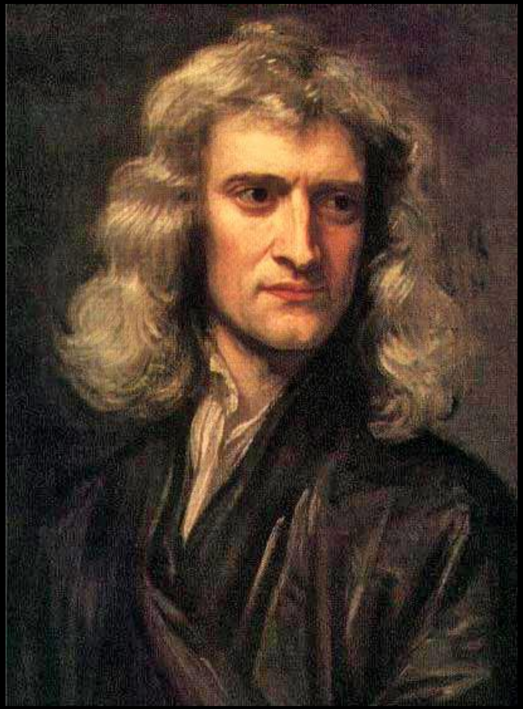
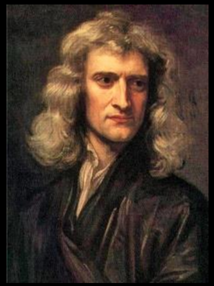
Counterpoint with history

Kepler (1571-1630)



Newton (1643-1727)



Laplace (1749-1827): this is the “world’s system”
All we have to do is to compute “f”, and the initial conditions

Why should we try to describe a problem with differential equations?
Should we? Can we? **After all…**

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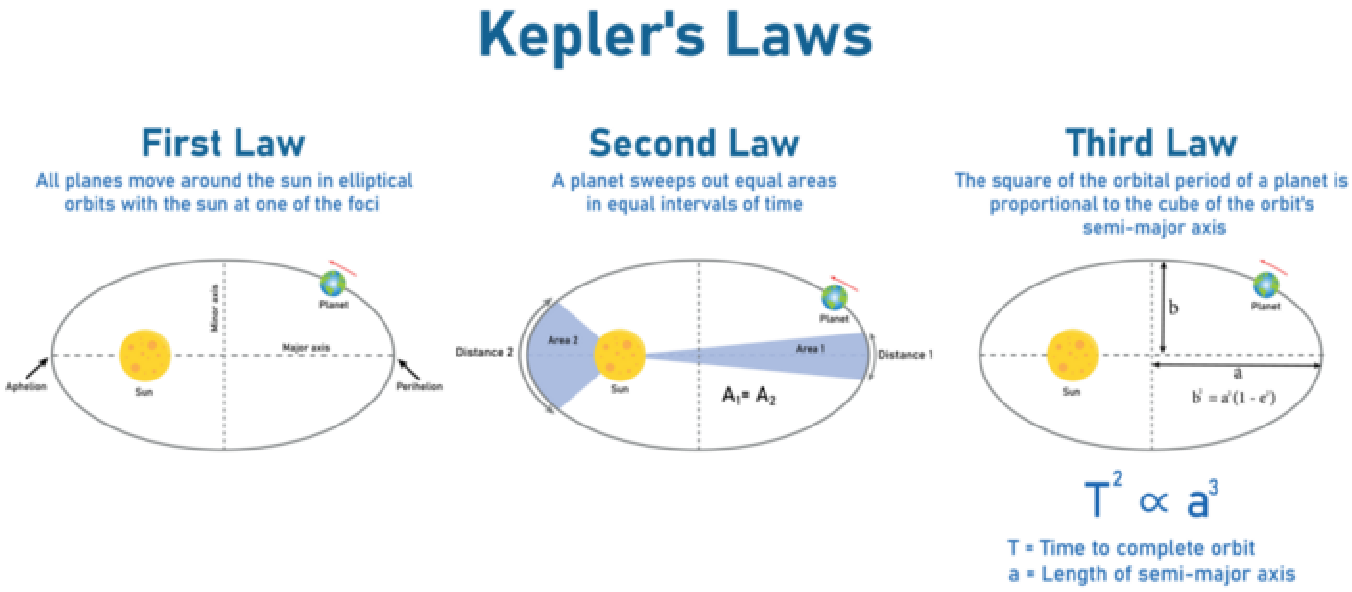
Should we? Can we? **After all…**

One of the biggest revolutions in science was data driven.
And the "conversation" between Kepler and Newton is
the first battle between data science and interpretable science



The variables in Kepler's laws are interpretable, but we have no interpretable mechanism relating them.

Newton’s approach gave us more

We define the velocity. $$\frac{dx}{dt} = v$$

$$\frac{dv}{dt} = \begin{cases} 0 & \text{In the absence of interctions with the universe (Galileo)} \\ \dfrac{1}{m} & \text{Multiplied by a functional form describing the interaction} \end{cases}$$

Some functions, fundamental ones

$$\boldsymbol{F} = \frac{-G\, m\, M}{{r_{12}}^{2}} \boldsymbol{e_{12}}$$

Others, phenomenological

$$F = -kx$$

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To perform a prediction, we have to have a model relating the variables

$$\boxed{\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= f(x, y) \end{aligned}}$$

More than a law, it is a program of how to do science

1. I measure a finite set of initial conditions.
2. I elucidate the law that prescribes the rate of variation of these variables (assuming that the law involves the values of the variables **at that moment**).
3. I integrate an ODE

It does NOT describe

1. Systems of equations with delays
2. Partial differential equations (although…)

$$\frac{dx(t)}{dt} = x(t) - \sigma x(t-\tau)^2$$

Even electromagnetism is excluded…

Dynamical systems
(in the framework of this course)

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(x) \qquad \boldsymbol{x} \in R^n$$

$$\boldsymbol{x}(t = 0) = \boldsymbol{x_0}$$

Under smoothness conditions for ***f(x)*** and ${}^{df}\!/_{dx}$ in a neighborhood of $\boldsymbol{x_0}$, the solution of the system exists and is unique. This means that each point (state) has a unique future (in geometrical terms, no auto-intersection of trajectories.

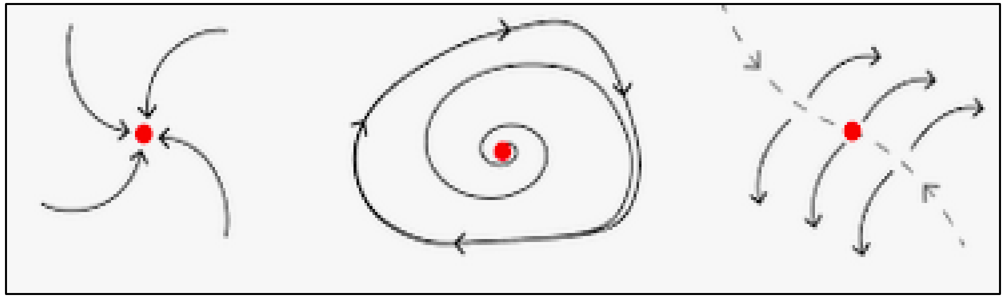
Dynamical systems
(in the framework of this course)

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(x) \qquad \boldsymbol{x} \in R^n$$

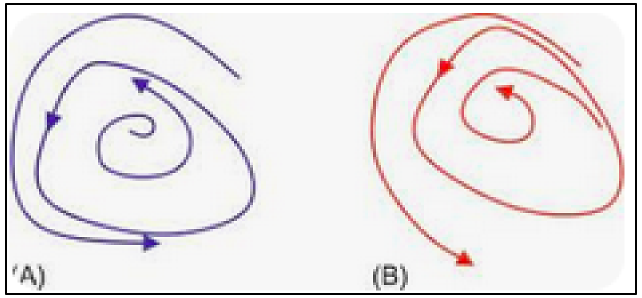
$$\boldsymbol{x}(t = 0) = \boldsymbol{x_0}$$

solving analytically these is usually impossible, therefore…

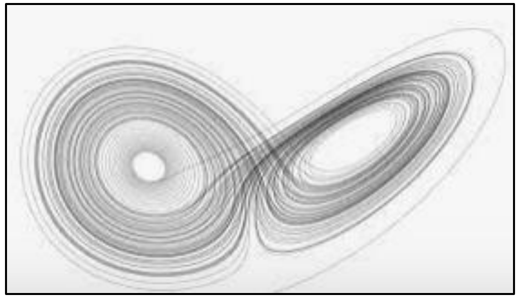
Fixed points



Limit cycles



Chaotic attractors



Example: the relaxation oscillator

$$\boxed{\frac{d^2x}{dt^2} + \mu + (x^2 - 1)\frac{dx}{dt} + kx = 0}$$

$$\frac{d}{dt}\Bigg(\underbrace{\frac{dx}{dt} + \frac{x^3}{3} - x}_{w}\Bigg) = -kx$$

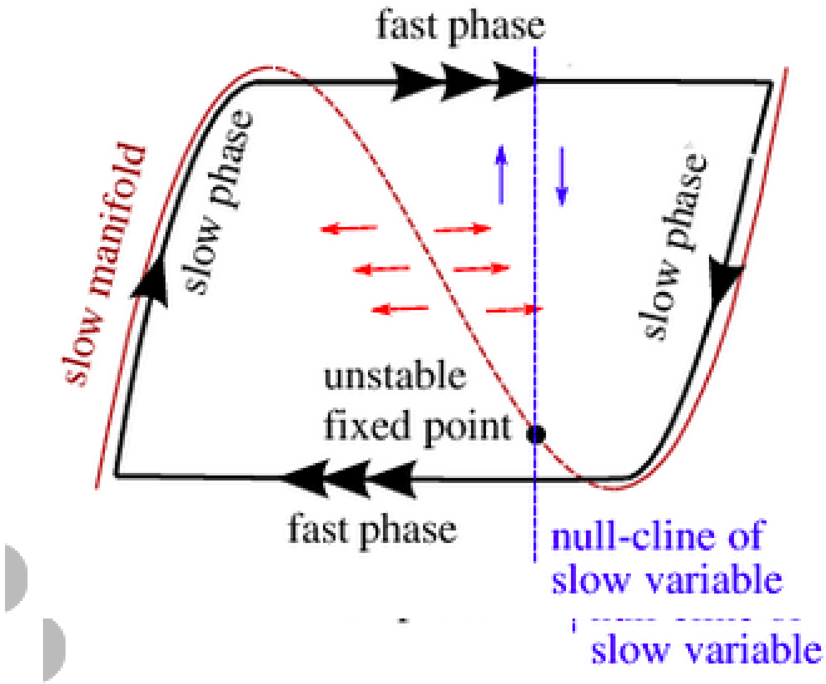
$$\tilde{w} \equiv w/\mu$$

$$\boxed{\begin{aligned} \frac{dx}{dt} &= \mu\left(\tilde{w} - \frac{1}{3}x^3 + u\right) \\ \frac{d\tilde{w}}{dt} &= -\frac{k}{\mu}(u) \end{aligned}}$$

Example: the relaxation oscillator

$$\boxed{\frac{d^2x}{dt^2} + \mu + (x^2 - 1)\frac{dx}{dt} + kx = 0}$$

If μ>> 1



$$\frac{dx}{dt} = \mu\left(\tilde{w} - \frac{1}{3}x^3 + u\right)$$

$$\frac{d\tilde{w}}{dt} = -\frac{k}{\mu}(u)$$

The “splash” affected all science, just as Laplace predicted

A “Newtonian” approximation to neuroscience

$$C\frac{dV}{dt} = I - g_K n^4 (V - E_K) - g_{Na} m^3 h (V - E_{Na}) - g_l (V - E_l)$$

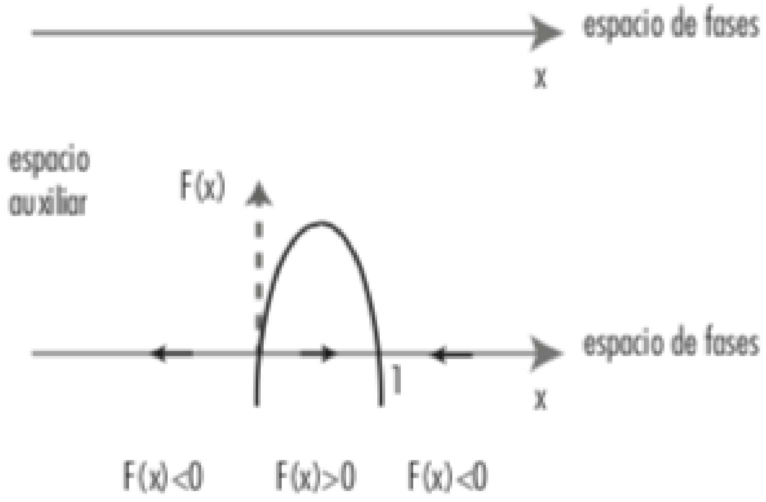
$$\frac{dn}{dt} = \alpha_n(V)(1-n) - \beta_n(V)n$$

$$\frac{dm}{dt} = \alpha_m(V)(1-m) - \beta_m(V)m$$

$$\frac{dh}{dt} = \alpha_h(V)(1-h) - \beta_h(V)h$$

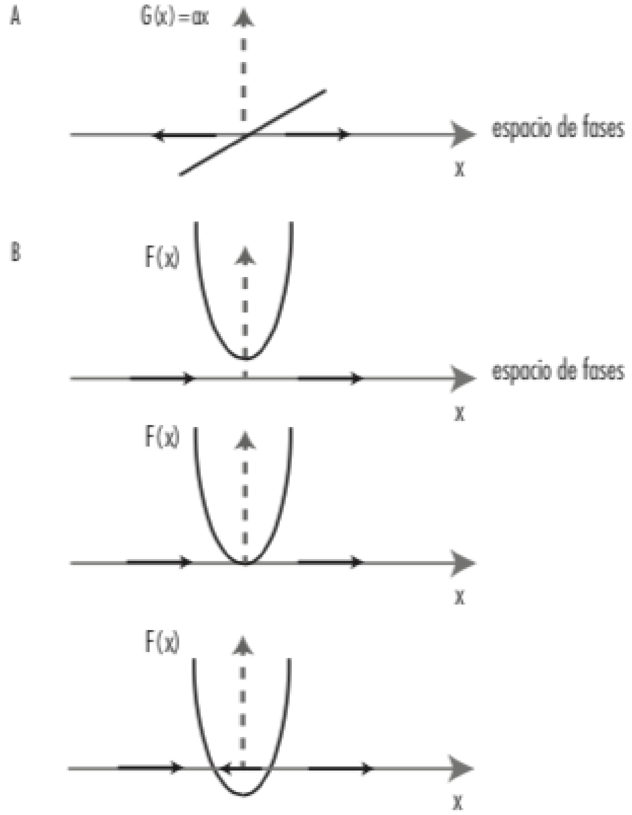
En que afecta al flujo el que el campo vector sea no lineal?

$$\boxed{\frac{dx}{dt} = ax\left(1 - \frac{x}{N}\right).}$$



La primera signatura de que un sistema es no lineal es la coexistencia de puntos estacionarios aislados

$$\frac{dx}{dt} = r + x^2$$
$$r \in \mathbb{R}$$



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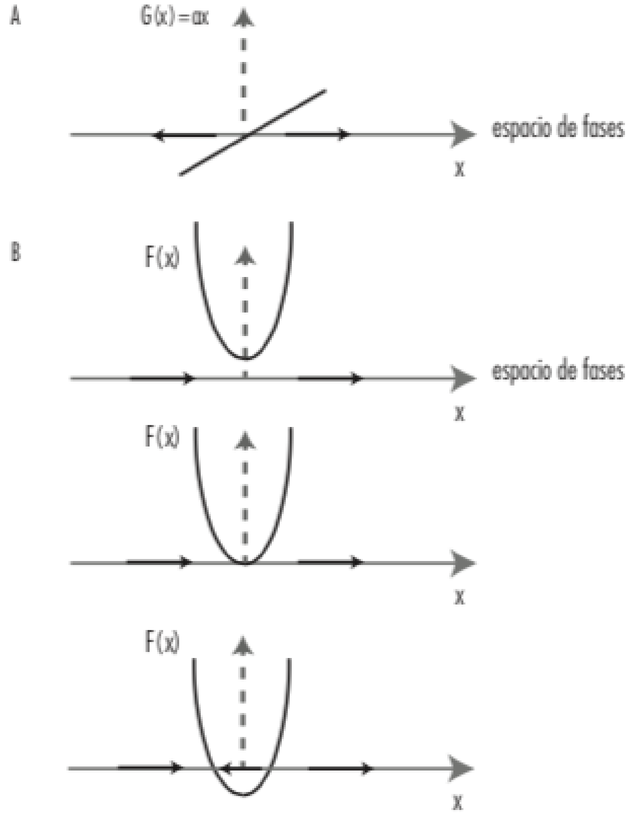
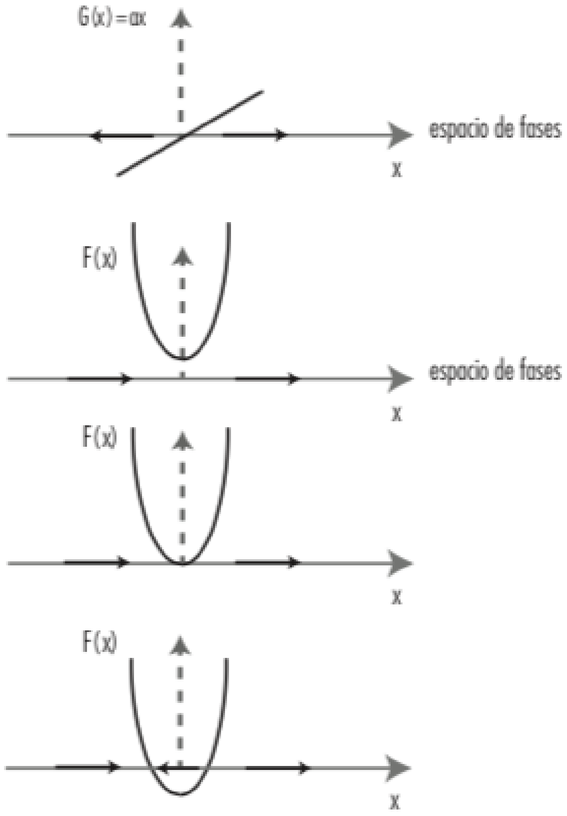
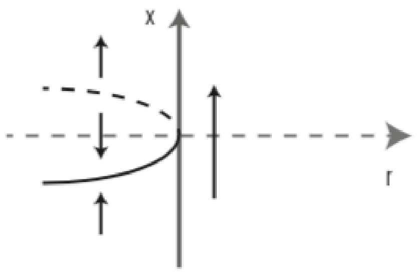


Dependiendo del parámetro, existe un numero diferente de puntos fijos. A tal cambio cualitativo se lo conoce como **<u>bifurcación</u>.** Notemos que un **sistema no lineal puede tener un numero mayor que uno de puntos fijos aislados.**

A

B



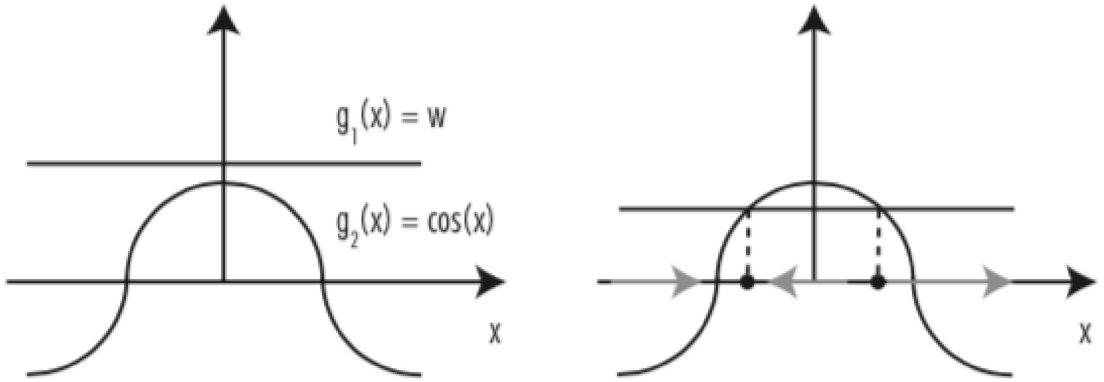


Nodo silla en ciclo limite

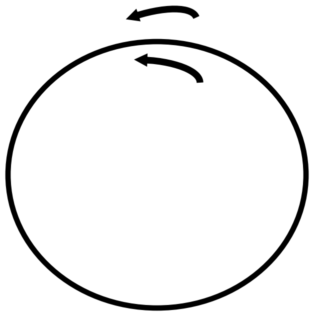
$$\frac{dr}{dt} = r(1 - r)$$

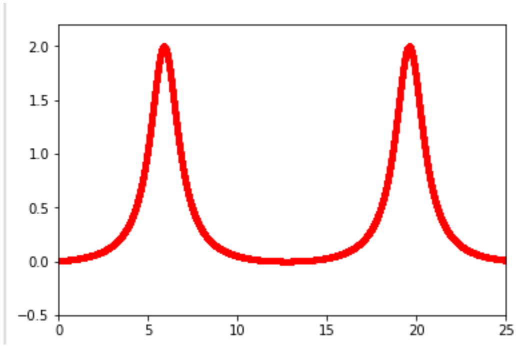
$$\frac{d\theta}{dt} = \omega - cos\theta$$

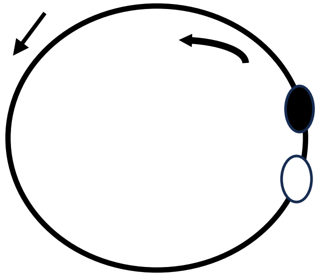
Notemos que para la parte angular,



O sea, cuando $\omega > 1$, el sistema oscila. Las oscilaciones nacen con periodo infinito.







Formas normales

$$\frac{dA_1}{dt} = A_2$$

$$\frac{dA_2}{dt} = (2\epsilon - {A_1}^2)A_2 - A_1$$

El primer termino de la segunda ecuación es una disipación (negativa si A_1 es pequeña, y positiva si A_1 es grande). El segundo termino, una restitución.

Este sistema de ecuaciones, entonces, da lugar a una oscilación de relajación, como hemos visto en la segunda clase.

Miremos el problema cerca de donde ocurrirán las bifurcaciones, esto es, cerca de $\epsilon \approx 0$. Para simplificar, definamos

$$X = A_1 + iA_2$$

Lo que da lugar a la ecuación en variable compleja:

$$\frac{dX}{dt} = -iX - \frac{1}{8}(X^3 + X^2\bar{X} - X\bar{X}^2 - \bar{X}^3)$$

Y la pregunta que nos vamos a hacer es si es posible encontrar un cambio de variables que elimine alguno de estos términos.

Proponemos un cambio de variables implícito, con todos los términos cúbicos posibles, o sea:

$$X = Z + \alpha Z^3 + \beta Z^2 \bar{Z} + \gamma Z \bar{Z}^2 + \delta \bar{Z}^3$$

$$(1 + 3\alpha Z^2 + 2\beta Z\bar{Z} + \gamma \bar{Z}^2)\frac{dZ}{dt} + (\beta Z^2 + 2\gamma Z\bar{Z} + 3\delta \bar{Z}^2)\frac{d\bar{Z}}{dt} =$$

$$= -i(Z + \alpha Z^3 + \beta Z^2\bar{Z} + \gamma Z\bar{Z}^2 + \delta \bar{Z}^3) - \frac{1}{8}(Z^3 + Z^2\bar{Z} - Z\bar{Z}^2 - \bar{Z}^3)$$

Y reemplazando $\frac{d\bar{Z}}{dt}$ por $i\bar{Z}$ (ya que va a entrar vía términos cúbicos, basta considerar la aproximación lineal), la ecuación queda:

$$\frac{dZ}{dt} = -iZ + \left(2\alpha - \frac{1}{8}\right)Z^3 + \left(-\frac{1}{8}\right)Z^2\bar{Z}$$

$$+\left(-2\gamma + \frac{1}{8}\right)Z\bar{Z}^2 + (-2\delta + \frac{1}{8})\bar{Z}^3$$

Notamos entonces, que podemos elegir α, γ, δ de modo que los términos con $Z^3, Z\bar{Z}^2, \bar{Z}^3$ se vayan, quedando

$$\frac{dZ}{dt} = -iZ - \frac{1}{8}\,Z^2\bar{Z}$$

Esta es una expresión minimal. No hay cambio de coordenada posible que elimine a este termino. Notemos que es un termino tal que, si pensamos que linealmente

$$Z \sim e^{-it}$$

$$Z^2 \bar{Z} \sim e^{-2it} \, e^{it} = e^{-it}$$

Mientras que los términos eliminados van como

$$Z^3 \sim e^{-3it}, Z\bar{Z}^2 \sim e^{it} \,, \bar{Z}^3 \sim e^{3it}$$

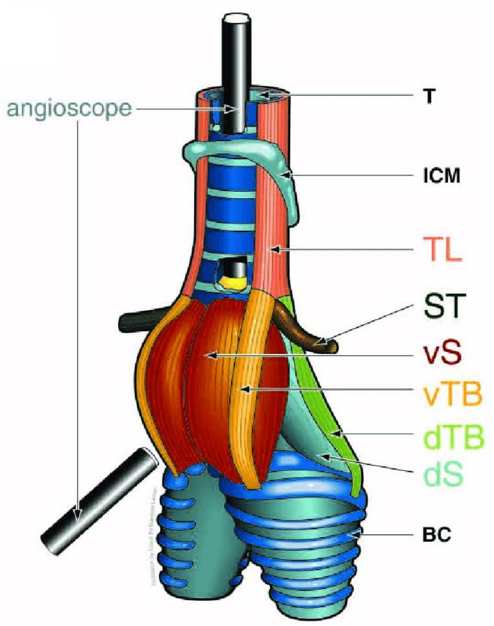
Por eso decimos que el cambio no lineal de coordenadas dejo solo al termino **resonante.**

Esta estrategia ya la habíamos esbozado, cuando presentamos, para bifurcaciones en las primeras dos clases, ecuaciones sencillas, paradigmáticas, que representaran las propiedades cualitativas de un flujo, y sus cambios ante cambios de los parámetros. Por ejemplo:

Example

¿How do birds generate their songs?

Example


angioscope
T
ICM
TL
ST
vS
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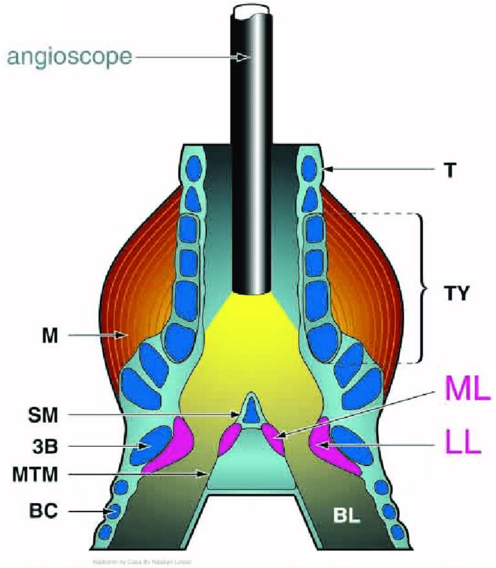
BC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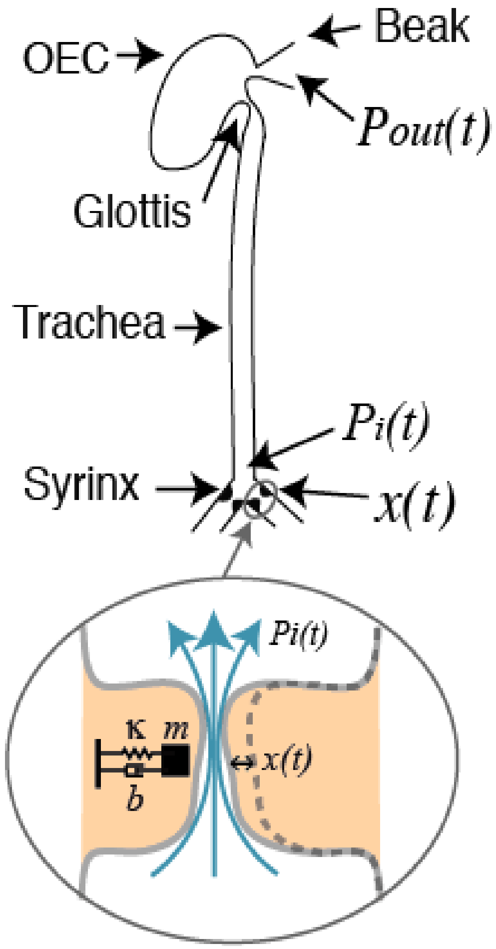
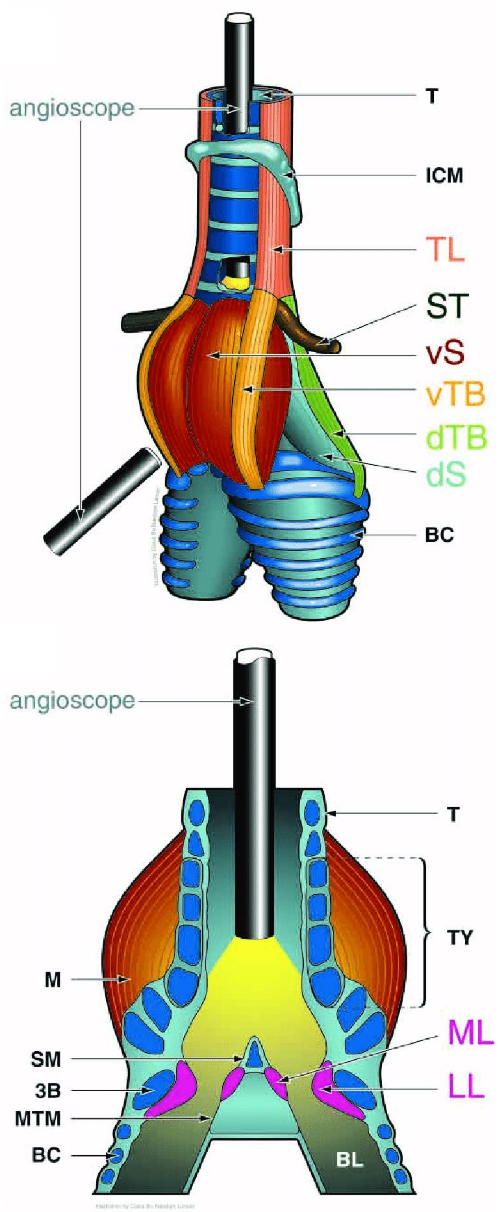
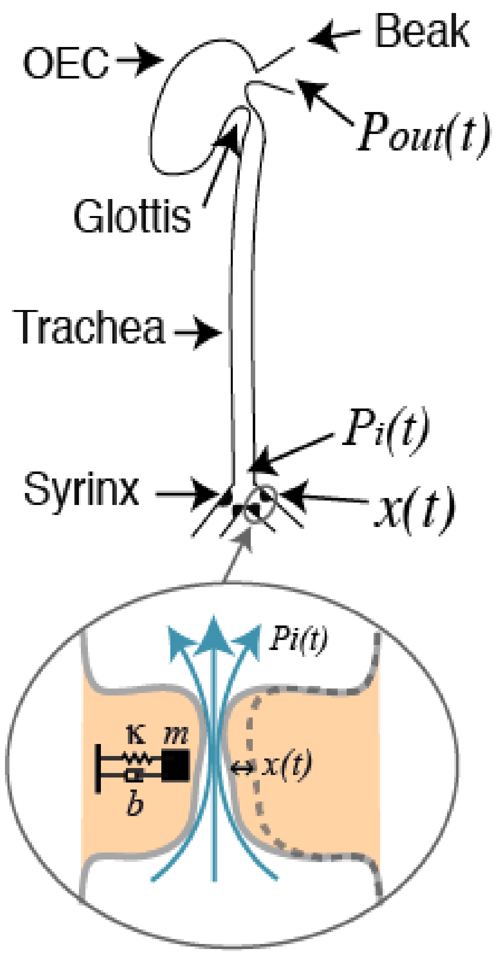
Example



Hypothesis



The program of physics (or interpretable science)

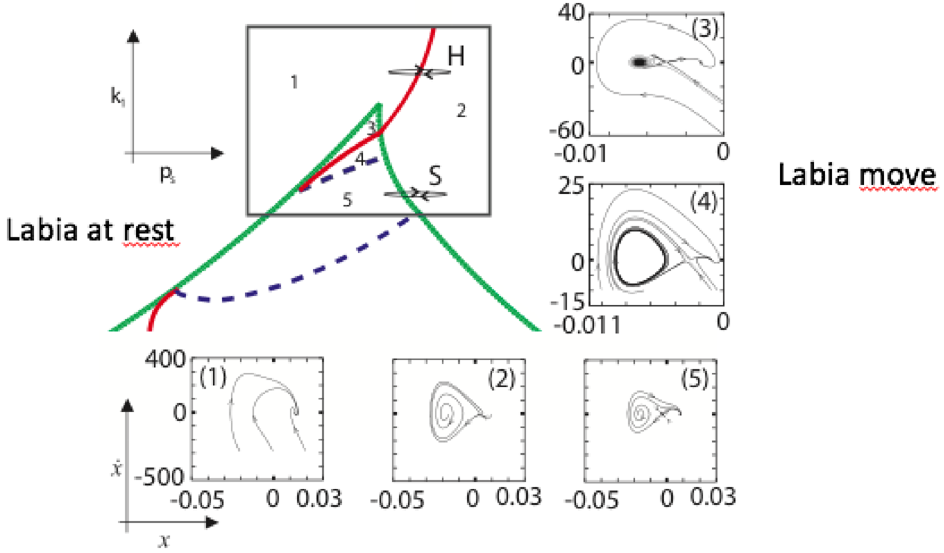


1. To identify the pertinent variables
And how to relate them through a model

$$
\begin{cases}
\dfrac{dx}{dt} = y, \\[2ex]
\dfrac{dy}{dt} = (1/m)\left[-k(x)x - b(y)y - cx^2y + a_{lab}p_s\left(\dfrac{\Delta a + 2\tau\tau}{a_{01} + x + \tau y} \right) \right].
\end{cases}
$$

2. To study the solutions for different parameters

Bifurcation diagram



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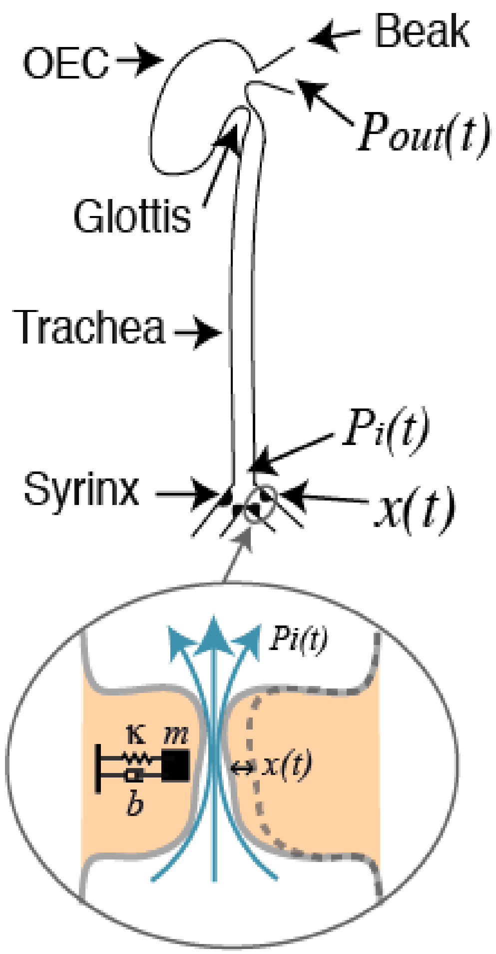
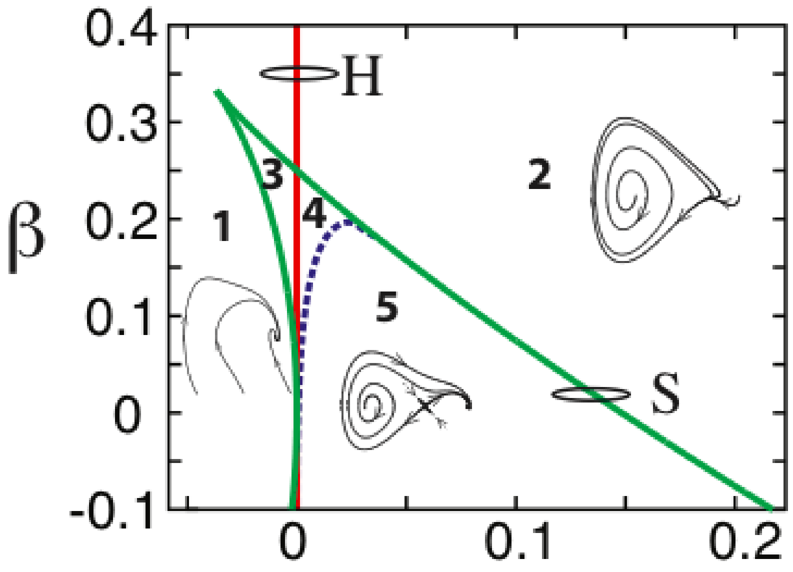


1. To find the simplest models

Normal form

$$
\begin{cases}
\dfrac{dx}{dt} = y \\
\dfrac{dy}{dt} = -a(t)g^2 - b(t)g^2x - g^2x^3 - gx^2y + g^2x^2 - gxy
\end{cases}
$$

2. To study the solutions of this simplified model



Normal form reduction

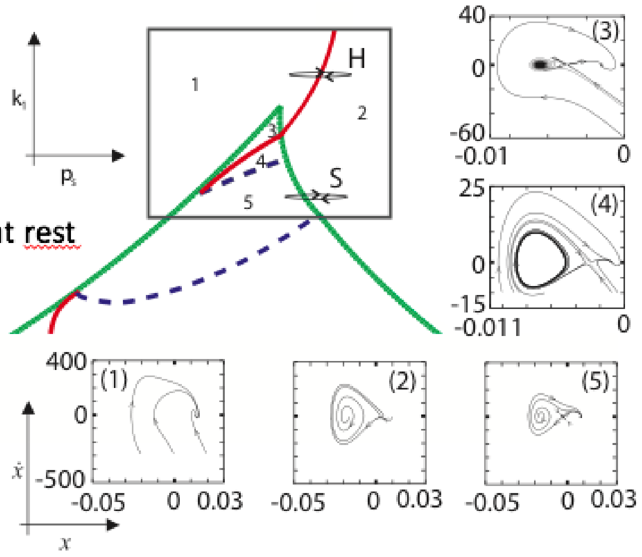
$$\begin{cases} \dfrac{dx}{dt} = y, \\ \dfrac{dy}{dt} = (1/m)\left[-k(x)x - b(y)y - cx^2y + a_{lab}p_s\left(\dfrac{\Delta a + 2\tau\tau}{a_{01} + x + \tau y}\right)\right]. \end{cases}$$

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -\alpha(t)\gamma^2 - \beta(t)\gamma^2 x - \gamma^2 x^3 - \gamma x^2 y + \gamma^2 x^2 - \gamma x y \end{cases}$$

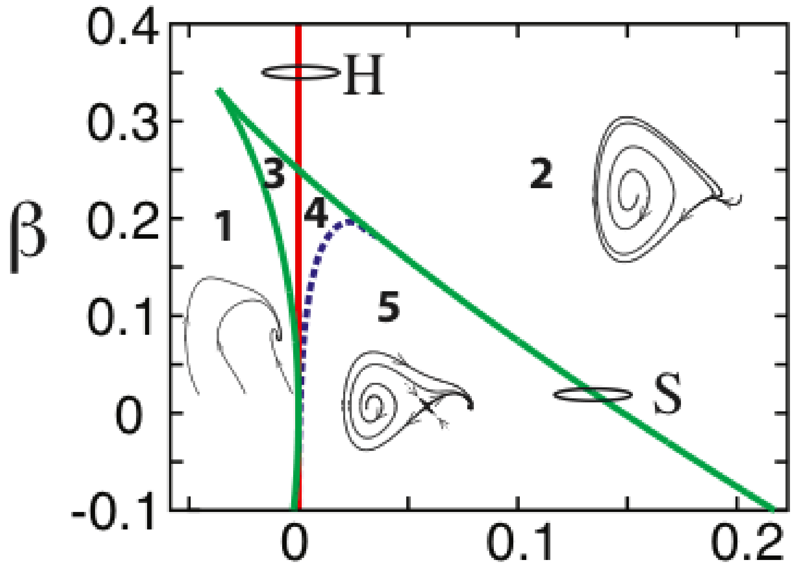
Algorithmic procedure that allows you to reduce your nonlinear problem to a simpler one, close to a linear singularity. This simpler model will have less parameters, and **has been probably studied**

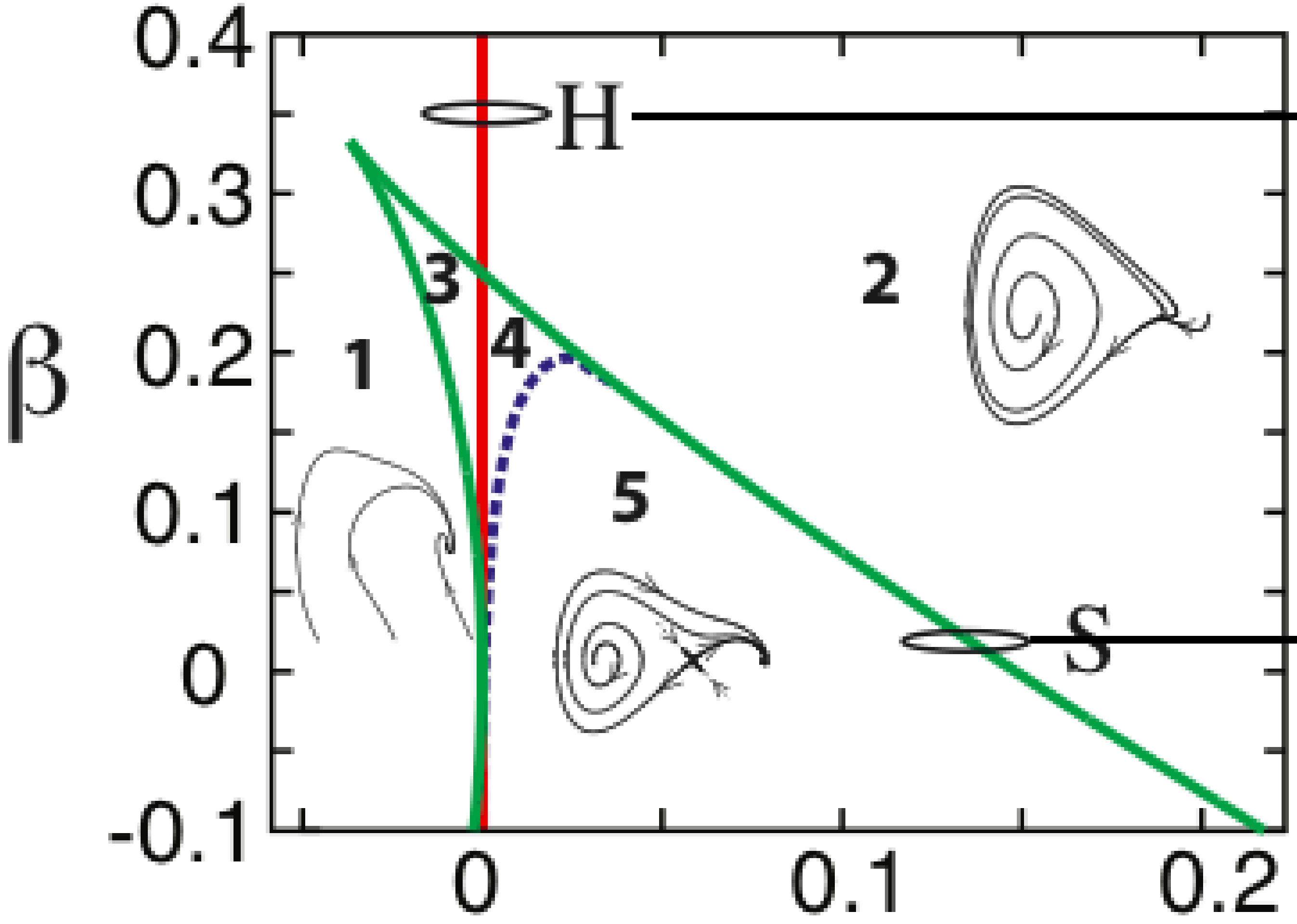
Normal form reduction

Labia at rest



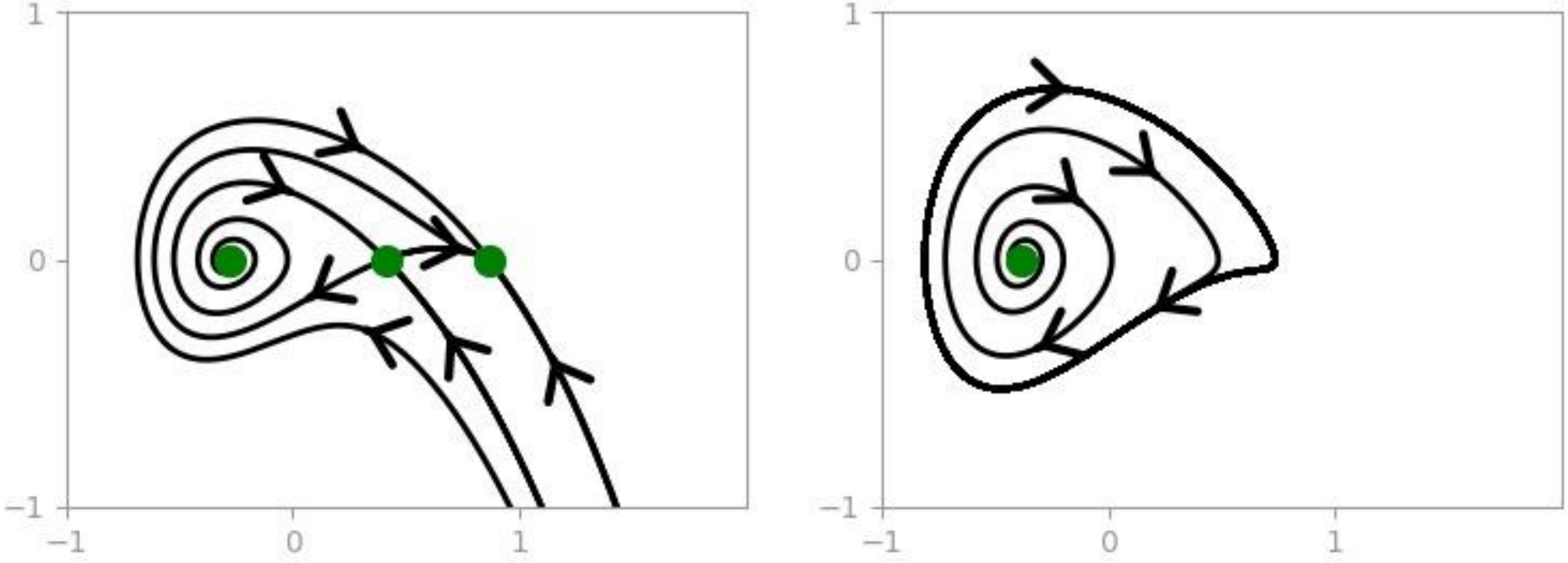
Labia move





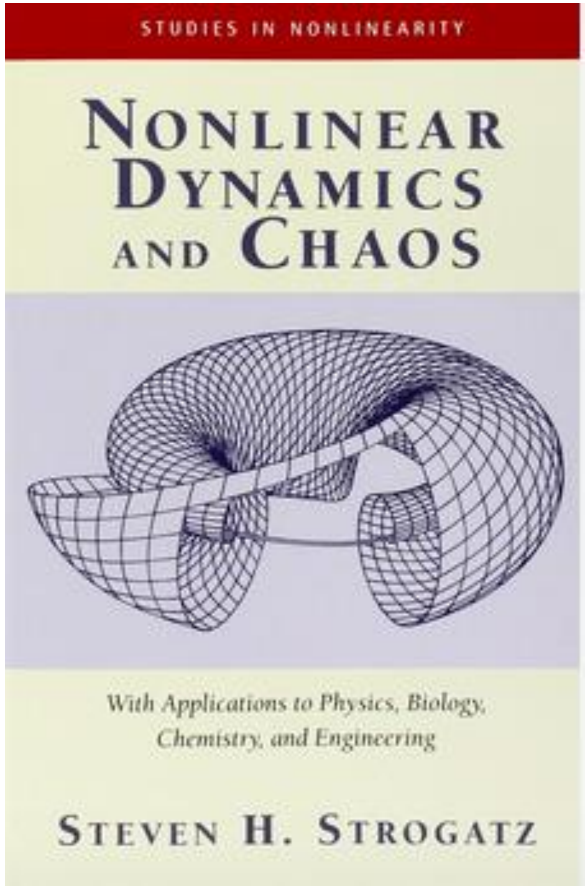
Oscillations born with finite frequency and a zero amplitude

Oscillations born with zero frequency and a finite amplitude



SNILC: one of the ways oscillations arise in 2d systems

How many? The good news… not that many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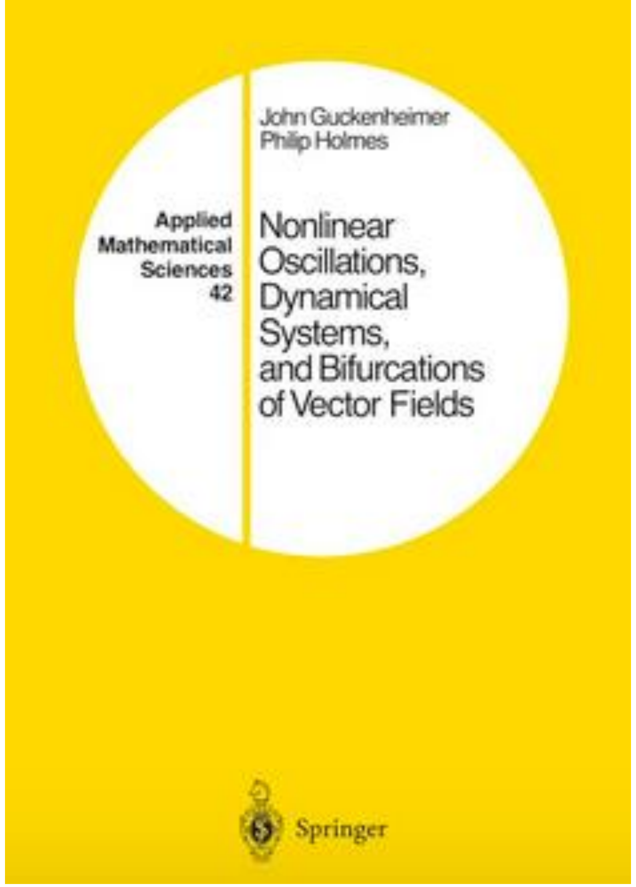
NONLINEAR DYNAMICS AND CHAOS
With Applications to Physics, Biology, Chemistry, and Engineering
STEVEN H. STROGATZ


John Guckenheimer
Philip Holmes
Applied Mathematical Sciences 42
Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields
Springer

Newton’s “mandate”

1. First, **understand** the **mechanisms**
2. Plug them into the machinery of **dynamical systems**
3. **Predict**

“Novelty” of the XX century:
qualitative theory of dynamical systems and
numerical integration as exploratory tool

But what if what we want to understand is extraordinarily complex?